DIFFERENTIAL EQUATIONS

EXERCISE 2.8

Problems solved by;

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NONHOMOGENEOUS EQUATIONS

A nonhomogeneous linear differentiation equation is written as $y'' + P(x)y' + q(x)y = r(x)$ ——— (1)

put $r(x) = 0$ in (1) we get the homogeneous eq as

$y'' + P(x)y' + q(x)y = 0$ ——— (2)

To solve (1) we have the following Theorem

THEOREM - 1

(a) The difference of the two solutions of (1) on some open interval I is a solution of (2) on I.

The sum of a solution of (1) and a solution of (2) on I respectively is a solution of (1) on I.

PROOF

(a) Let In (1) $R.H.S = L(y)$

& let y and $\bar{y}$ be any solutions of (1) on I.

Then $L(y) = r(x)$ and $L(\bar{y}) = r(x)$. $\because (y-\bar{y})' = y' - \bar{y}'$ ...

$\therefore L(y-\bar{y}) = L(y) - L(\bar{y}) = r(x) - r(x) = 0$.

Similarly, for y of (1) and any solution y^* of (2) on I

$L(y + y^*) = L(y) + L(y^*) = r(x) + 0 = r(x)$.

GENERAL SOLUTION OF — (1)

A general solution of the non-homogeneous linear equation (1) on some open interval I is a solution of the form.

$y(x) = y_h(x) + y_p(x)$ ——— (3)

PARTICULAR SOLUTION: A particular solution of (1) on I is obtained from (3) by assigning specific values to the arbitrary constants c_1 and c_2 in $y_h(x)$

⇒ A general Solution of (1) includes all solutions

THEOREM # 2

Suppose that the coefficients and $r(x)$ in (1) are continuous on some open interval I, Then every sol of (1) on I is obtained by assigning suitable values to the arbitrary constants in a general Solution (3) of (1) on I.

PRACTICAL CONCLUTIONS: To solve the nonhomogeneous equations as (1), or an initial value problem for (1), we have to solve the homogeneous equations (2) and find any particular Solution y_p of (1)

EXAMPLE

Solve the initial value problem

$$y'' + 2y' + 101y = 10.4e^x, \quad y(0) = 1.1, \; y'(0) = -0.9 \quad \text{—(1)}$$

step # 01 The general sol of the homogeneous equation:

The characteristic eq is

$$\lambda^2 + 2\lambda + 101 = 0$$

$$\Rightarrow \lambda = \frac{-2 \pm \sqrt{4 - 4(101)}}{2} = -1 \pm \sqrt{1 - 101}$$

$$\Rightarrow \lambda = -1 \pm 10i.$$

Step # 02

General Solution of non-homogeneous equation

For this we need any particular solution y_p of the non-homogeneous equation. Since e^x on the right has the derivatives e^x and e^x, we type.

Step 2

$y_p = ce^x$.

$\Rightarrow y'_p = ce^x$ & $y''_p = ce^x$. putting in (1)

$\Rightarrow ce^x + 2ce^x + 101ce^x = 10.4e^x \Rightarrow (1+2+101)ce^x = 10.4e^x$

$\Rightarrow ce^x(104) = 10.4e^x$.

$\Rightarrow c = \frac{10.4}{104} = 0.1$

$\therefore y_p = 0.1e^x$.

and the general solution of (1) is.

$y = y_h + y_p = e^{-x}(A\cos(10x) + B\sin(10x)) + 0.1e^x$. — (3)

Step # 03 Particular Solution

Applying initial condition to find the P.S

we have from 1st initial condition.

$1.1 = A\cos 0 + 0.1 \Rightarrow A = 1$

Also

$y' = -e^{-x}(A\cos(10x) + B\sin(10x)) + e^{-x}(-10A\sin 10x + 10B\cos 10x) + 0.1e^x$.

Applying 2nd initial condition we get

$-0.9 = -A\cos 0 + 10B\cos 0 + 0.1$

$\Rightarrow -A + 10B = -0.1 - 0.9$

$\Rightarrow 10B = -1 + 1 \Rightarrow B = 0/10 = 0$

hence the particular solution is.

$y = e^{-x}\cos(10x) + 0.1e^x$.

PARTICULAR & GENERAL SOLUTIONS

Verify that y_p is a solution of the given diff. equation and find a general solution.

(1) $y'' - y = 8e^{-3x}$ —(1), $y_p = e^{-3x}$

Sol

$y_p = e^{-3x} \Rightarrow y' = -3e^{-3x} \Rightarrow y'' = 9e^{-3x}$.

putting in (1)

$\Rightarrow 9e^{-3x} - e^{-3x} = 8e^{-3x}$

verified

To find general solution first we have to find the solution of corresponding non-homogeneous differential equation.

The characteristic equation for that is.

$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$\therefore y_h = c_1 e^{-x} + c_2 e^{x}$.

$\therefore$ the general sol of (1) is.

$y = y_h + y_p = c_1 e^{-x} + c_2 e^{x} + e^{-3x}$

Ans

(2) $y'' - y = 8e^{-3x}$ —(1), $y_p = e^{-3x} - 3e^{x}$

Sol $y_p = e^{-3x} - 3e^{x} \Rightarrow y_p' = -3e^{-3x} - 3e^{x}$

$\Rightarrow y_p'' = 9e^{-3x} - 3e^{x}$.

putting in (1)

$\Rightarrow 9e^{-3x} - 3e^{x} - e^{-3x} - 3e^{x} = 8e^{-3x}$

verified.

To find G.S of (1).

i.e $y = y_h + y_p$.

For y_h, we have the characteristic eq.

$\lambda^2 - 1 = 0$

$\Rightarrow \lambda = \pm 1$.

$\Rightarrow y_h = c_1 e^{-x} + c_2 e^{x}$.

$\therefore$ The G.S of (1) is.

$y = y_h + y_p = c_1 e^{-x} + c_2 e^{x} + e^{-3x} - 3e^{x}$ Ans

Sol $y_p = 2x^2 - 6x + 7 \Rightarrow y_p' = 4x - 6 \Rightarrow y_p'' = 4$.

putting in (1)

$\Rightarrow 4 + 12x - 18 + 4x^2 - 12x + 14 = 4x^2$

verified

Q3

The g.s. is

$y = y_h + y_p$.

For y_h, consider the corresponding homogeneous linear eq

The characteristic equation is

$\lambda^2 + 3\lambda + 2 = 0$.

$\Rightarrow \lambda^2 + 2\lambda + \lambda + 2 = 0 \Rightarrow \lambda(\lambda + 2) + 1(\lambda + 2) = 0$.

$\Rightarrow \lambda_1 = -1 \quad , \lambda_2 = -2$.

hence.

$y_h = c_1 e^{-x} + c_2 e^{-2x}$

and the g.s of (1) i.e.

$y = c_1 e^{-x} + c_2 e^{-2x} + 2x^2 - 6x + 7$

Ans

(4) $y'' - 2y' + 5y = 5x^3 - 6x^2 + 6x$, $y_p = x^3$. (1)

Sol $y_p = x^3 \Rightarrow y_p' = 3x^2 \Rightarrow y_p'' = 6x$ putting in (1)

$\Rightarrow 6x - 6x^2 + 5x^3 = 5x^3 - 6x^2 + 6x$.

verified

Now $y = y_h + y_p$.

For y_h. we have the the characteristic eq of the corresponding homogeneous differential Eq as.

$\lambda^2 - 2\lambda + 5 = 0$.

$\Rightarrow \lambda = \dfrac{2 \pm \sqrt{4 - 4(5)}}{2} = 1 \pm 2i$.

hence. $y_h = e^x (A\cos 2x + B\sin 2x)$.

and

$y = e^x (A\cos 2x + B\sin 2x) + 5x^3 - 6x^2 + 6x$

Ans

$y_p = -\cos 2x.$

Sol $y_p = -\cos 2x \Rightarrow y'_p = 2\sin 2x \Rightarrow y''_p = 4\cos 2x$

Q5 putting in ①

$\Rightarrow 4\cos 2x + 6\sin 2x + 4\cos 2x = 8\cos 2x + 6\sin 2x.$

verified.

Now

$y = y_h + y_p.$

For y_h we have the characteristic eq of the corresponding homogeneous differential eq as.

$D^2 + 3D - 4 = 0$

$\Rightarrow D^2 + 4D - D - 4 = 0 \Rightarrow D(D+4) - 1(D+4) = 0.$

$\Rightarrow D = -4, \quad D = 1$

hence the general solution of homogeneous eq is

$y_h = c_1 e^{-4x} + c_2 e^{x}$

and G.S of nonhomogeneous eq is.

$y = c_1 e^{-4x} + c_2 e^{x} - \cos 2x$ Ans

⑥ $(D^2 - 4D + 4)y = e^{x}\cos x, \quad y_p = -\frac{1}{8} e^{x}\sin x$

(Solve yourself)

⑦ $(D^2+1)y = -x^{-2} + \ln \pi x, \quad y_p = \ln \pi x.$ ①

Sol $y_p = \ln \pi x \Rightarrow y'_p = \pi \frac{1}{\pi x} = \frac{1}{x} = x^{-1}.$

$\Rightarrow y''_p = -x^{-2}.$ putting in ① we get.

$-x^{-2} + \ln \pi x = -x^{-2} + \ln \pi x.$ verified

Now

$y = y_h + y_p.$ —— ②

for y_h we have the characteristic eq of the corresponding homogeneous differential eq as

$\lambda^2 + 1 = 0.$

$\Rightarrow \lambda = \pm i$

hence $y_h = A\cos x + B\sin x.$ putting in ② we get

⑧ $(8D^2 - 6D + 1)y = 6\cosh x$ —① , $y_p = \frac{1}{5}e^{-x} + e^x$

Sol $y_p = \frac{1}{5}e^{-x} + e^x \Rightarrow y'_p = -\frac{1}{5}e^{-x} + e^x \Rightarrow y''_p = \frac{1}{5}e^{-x} + e^x$.

Q8

putting in ①

$\Rightarrow \frac{8}{5}e^{-x} + 8e^x + \frac{6}{5}e^{-x} - 6e^x + \frac{1}{5}e^{-x} + e^{+x} = 6\cosh x$

$\Rightarrow \frac{8+6+1}{5}e^{-x} + (8-6+1)e^x + e^{-x} = 6\cosh x$.

$\Rightarrow 3e^{-x} + e^{+x} + 2e^x = 6\cosh x$.

$\Rightarrow 3e^{-x} + 3e^x = 6\cosh x$.

$\Rightarrow 3(e^x + e^{-x}) = 6\cosh x$.

$\Rightarrow 6\left(\frac{e^x + e^{-x}}{2}\right) = 6\cosh x$

$\Rightarrow 6\cosh x = 6\cosh x$ verified

Now the G.S of ① is given by.

$y = y_h + y_p$. —②

For y_h, we have the characteristic eq of the corresponding homogeneous differential equation as.

$8\lambda^2 - 6\lambda + 1 = 0$

$\Rightarrow \lambda = \frac{6 \pm \sqrt{36 - 8(4)}}{16} = \frac{6 \pm \sqrt{36-32}}{16} = \frac{6 \pm 2}{16}$

$\Rightarrow \lambda = 1/2$, $\lambda = 1/4$.

hence $y_h = c_1e^{x/2} + c_2e^{x/4}$. putting in ②

$\Rightarrow y = c_1e^{x/2} + c_2e^{x/4} + \frac{1}{5}e^{-x} + e^x$. Ans

INITIAL VALUE PROBLEMS

In prob. 9–15, verify that y_p is a solution of the given equation. Solve the initial value problem

⑨ $y'' + y = 2x$, $y(0) = -1$, $y'(0) = 8$, $y_p = 2x$

$y = y_h + y_p$ —①

For y_h, we have the characteristic eq of the

$\therefore \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$.

hence $y_h = A\cos x + B\sin x$. putting in (1) we get

$y = A\cos x + B\sin x + 2x$, (the general solution)

Also $y' = -A\sin x + B\cos x + 2$.

Applying Initial conditions we have

$-1 = A$, $\quad 8 = B + 2 \Rightarrow B = 6$.

$\therefore y = -\cos x + 6\sin x + 2x$. Ans

Q#10 Solve Yourself.

Q#11 $y'' - y = 2e^x$; $y(0) = -1$, $y'(0) = 0$; $y_p = xe^x$.

The general sol is given by

$y = y_h + y_p$ —— (1).

For y_h, we have the characteristic eq of the corresponding homogeneous eq as

$\lambda^2 - 1 = 0 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

$\therefore y_h = c_1 e^x + c_2 e^{-x}$. putting in (1)

$\Rightarrow y = c_1 e^x + c_2 e^{-x} + xe^x$.

Also $y' = c_1 e^x - c_2 e^{-x} + xe^x + e^x$.

Applying Initial conditions we have.

$-1 = c_1 + c_2 \Rightarrow c_1 + c_2 = -1$ —— (3)

& $0 = c_1 - c_2 + 1 \Rightarrow c_1 - c_2 = -1$ —— (4).

adding (3) and (4) we get

$2c_1 = -2 \Rightarrow c_1 = -1$.

and hence $c_2 = 0$.

So

$y = -e^x + xe^x$ Ans

$y(0) = 1.8$, $y'(0) = 5$; $y_p = 3x\cos 2x$.

we have the general sol as.

Q12 $y = y_h + y_p$ —— (2)

For y_h, we have the characteristic eq of the corresponding homogeneous linear eq as.

$D^2 + 4 = 0 \Rightarrow D = \pm 2i$.

hence.

$y_h = A\cos 2x + B\sin 2x$ putting in (2)

$\Rightarrow y = A\cos 2x + B\sin 2x + 3x\cos 2x$.

Apply initial conditions and get p.s yourself.

Q#13

$(x^2D^2 - 3xD + 3)y = 3\ln x - 4$, $y(1) = 0$, $y'(1) = 1$

& $y_p = \ln x$

we have the general solution as.

$y = y_h + y_p$. —(1)

for, y_h, we have the characteristic eq of the corresponding homogeneous linear eq as.

$m^2 + (-3-1)m + 3 = 0$.

$\Rightarrow m^2 - 4m + 3 = 0$

$\Rightarrow m^2 - 3m - m + 3 = 0 \Rightarrow m(m-3) - 1(m-3) = 0$

$\Rightarrow m = 1$, $m = 3$.

hence the general solution

$y_h = c_1x + c_2x^3$. putting in (1)

$\Rightarrow y = c_1x + c_2x^3 + \ln x$ —(2)

Also $y' = c_1 + 3c_2x^2 + 1/x$ —(3)

Applying initial conditions we get.

$0 = c_1 + c_2 + \ln 1 \Rightarrow c_1 + c_2 = 0$ —(4)

& $1 = c_1 + 3c_2 + 1 \Rightarrow c_1 + 3c_2 = 0$.

$\Rightarrow 2c_2 = 0$, $\Rightarrow c_2 = 0$.

& $c_1 = 0$. putting in (2)

$\Rightarrow y = \ln x$ Ans

$y(1) = 2+3e \quad ; \quad y'(1) = 3e \;;\; y_p = -e^{-2x}\ln x$.

Sol We have the general sol as.

Q14 $y = y_h + y_p$ —①.

For y_h, we have the characteristic eq of the corresponding homogeneous eq as.

$m^2 + (-2-1)m + 2 = 0$.

$\Rightarrow m^2 - 3m + 2 = 0$.

$\Rightarrow m^2 - 2m - m + 2 = 0 \Rightarrow m(m-2) - 1(m-2) = 0$.

$\Rightarrow m = 1 \;, m = 2$.

$\therefore y_h = c_1 x^1 + c_2 x^2$. putting in ①

$\Rightarrow y = c_1 x + c_2 x^2 - e^{-2x}\ln x$ —②.

Applying Initial conditions we have.

$2+3e = c_1 + c_2 - e^{-2}\ln 1$

$\Rightarrow c_1 + c_2 = 2+3e$ —③

Also $y' = c_1 + 2c_2 x + 2e^{-2x}\ln x - \frac{e^{-2x}}{x}$.

$\Rightarrow 3e = c_1 + 2c_2$.

$\Rightarrow c_1 + 2c_2 = 3e$ —④

subtracting ④ from ③ we get.

$-c_2 = 2 + 3e - 3e$.

$\Rightarrow c_2 = -2$.

and Also By putting in ③.

$\Rightarrow c_1 = 4+3e$. putting in ②.

$\Rightarrow y = (4+3e)x - 2x^2 - e^{-2x}\ln x$.

Ans

Check your Answer yourself

Q#15 $(D^2+4D+4)y = \frac{e^{-2x}}{x^2}$ — ①, $y(1) = 1/e^2$, $y'(1) = -2/e^2$.

Q15 $y_P = -e^{-2x}\ln x$

Sol we have the general. sol of ① as

$y = y_h + y_p$ — ②

For, y_h we have the ~~solution~~ characteristic eq of the corresponding homogeneous linear diff equation as

$D^2+4D+4=0 \Rightarrow D = \frac{-4 \pm \sqrt{16-16}}{2} = -2.$

$\therefore y_h = (c_1 + c_2 x)e^{-2x}.$ putting in ② we have

$y = (c_1 + c_2 x)e^{-2x} - e^{-2x}\ln x.$

$\Rightarrow y = e^{-2x}(c_1 + c_2 x - \ln x)$ — ③

Also $y' = -2e^{-2x}(c_1 + c_2 x - \ln x) + e^{-2x}(c_2 - 1/x)$ — ④.

Applying initial conditions we get.

$1/e^2 = e^{-2}(c_1 + c_2) \Rightarrow c_1 + c_2 = 1$ — ⑤

and $-2/e^2 = -2e^{-2}(c_1 + c_2) + e^{-2}(c_2 - 1).$

$\Rightarrow -2/e^2 = -2e^{-2}(c_1 + c_2 + c_2/2 - 1/2)$

$\Rightarrow 1 = c_1 + \frac{3}{2}c_2 - 1/2 \Rightarrow c_1 + \frac{3}{2}c_2 = 3/2.$

$\Rightarrow 2c_1 + 3c_2 = 3$ — ⑥

xing ⑤ with ② and subtracting from ⑥

$$\begin{array}{r} \Rightarrow \cancel{2c_1} + 2c_2 = 2. \\ \cancel{2c_1} + 3c_2 = 3 \\ \hline c_2 = 1 \end{array}$$

putting in ⑤ $\Rightarrow c_1 = 0$. puting in ③

$\Rightarrow y = e^{-2x}(x - \ln x).$ Ans