DIFFERENTIAL EQUATIONS

EXERCISE 2.7

Problems solved by;

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EXISTENCE & UNIQUENESS.

THEORY, WRONSKIAN

In this case, we give a general Solution for homogeneous linear equation.

$$y'' + P(x)y' + q(x)y = 0 \quad \text{--- (1)}$$

with continuous, but otherwise arbitary variable co-efficients P and q. They will concern the existence of general solution.

$$y = c_1y_1 + c_2y_2 \quad \text{--- (2)}$$

of (1) as well as initial value problem consisting of differential equation (1) and two initial conditions

$$y(x_0) = k_0 \ , \ y'(x_0) = K_1 \quad \text{--- (3)}$$

with given x_0, k_0 and K_1

Clearly, no such theory was needed for constant coefficients or Euler, cauchy Equation because everything came out explicitly from our calculations.

Central to Present discussion in the following theorem

EXISTENCE & UNIQUENESS THEOREM FOR INITIAL VALUE PROBLEMS

If $P(x)$ and $q(x)$ are continuous functions on some open interval I and x_0 is in I, then the Initial value problem consisting of (1) and (3) has a unique solution on I

LINEAR INDEPENDENCE OF THE SOLUTIONS, WRONSKIAN

Theorem - 1 will imply very much imp

Independent Solutions. In this section we use wronskian determinant or briefly, the Wronskian of the two solutions y_1 and y_2 of (1) defined by.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

THEOREM- 2

(Linear Dependence & Independence of the solutions)

Suppose that (1) has continuous coefficients $p(x)$ and $q(x)$ on an open interval I, Then two solution y_1 and y_2 of (1) on I are linearly dependent on I. Iff their wronskian W is zero at some x_0 on I, Furthermore, If $W=0$ for $x=x_0$, then $W=0$ on I, Hence there is an x_1 in I at which W is not zero, Then y_1 and y_2 are linearly dependent on I.

$a = -(m_1 - m_2)$

$b = m_1 m_2$

$x^2y'' + (a+1)xy' + by = 0.$

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(10) x^5, x^{-5}

here $m_1 = 5$ and $m_2 = -5$

hence

$a = -(5-5) = 0$

$b = 5 \times -5 = -25$.

So the (cauchy equation) 2nd order L.D.E is.

$x^2y'' + (a+1)xy' + by = 0$

$\Rightarrow x^2y'' + xy' + 25y = 0.$

Also

$\left| \begin{matrix} 5x^{-1} & -5x \end{matrix} \right| = -5x^{-1} - 5x^{-1}$

$= -10x^{-1} = -10/x$.

hence solutions are L.I for

(11) $x^2, x^2\ln x$. Here $m_1 = m_2 = 2$

$a = -(2+2) = -4$. & $b = 2 \times 2 = 4$.

So the Eq is

$x^2y'' + (a+1)xy' + by = 0$.

$\Rightarrow x^2y'' - 3xy' + 4y = 0$

Check

$m^2 + (-3-1)m + 4 = 0$

$\Rightarrow m^2 - 4m + 4 = 0$

$\Rightarrow m^2 - 2m - 2m + 4 = 0$.

$\Rightarrow m(m-2) - 2(m-2) = 0$

$\Rightarrow m = 2$. hence our Eq is correct.

Now

$W = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & 2x\ln x + x \end{vmatrix} = 2x^3\ln x + x^3 - 2x^3\ln x$

$= x^3$.

Hence the sol are L.I iff $x \neq 0$.

Ans

(12) $\cosh 2x, \sinh 2x$.

Since they are L.I.

$\therefore y = c_1\cosh 2x + c_2\sinh 2x$

$\Rightarrow y' = 2c_1\sinh 2x + 2c_2\cosh 2x$.

$\Rightarrow y'' = 4c_1\cosh 2x + 4c_2\sinh 2x$

$\Rightarrow y'' = 4y$.

Check $\Rightarrow y'' - 4y = 0$

$\Rightarrow \lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2$.

$\therefore \Rightarrow y_1 = c_1e^{2x} + c_2e^{-2x}$

$\Rightarrow$ ~~$y_1 = c_1\left(\frac{e^{x}+e^{x}}{2}\right) + c_2\left(\frac{e^{-x}+e^{-2}}{2}\right)$~~

$\Rightarrow y_1 = \left(\frac{c_1 + c_2 - (c_2 - c_1)}{2}\right)e^{2x} + \left(\frac{c_1 + c_2 + c_2 - c_1}{2}\right)e^{-2x}$

$\Rightarrow y_1 = \frac{c_1 + c_2}{2}(e^{2x} + e^{-2x}) + \frac{c_1 - c_2}{2}(e^{2x} - e^{-2x})$

(13) $x^2, x^{1/2}$. Here $m_1 = 2$ and $m_2 = 1/2$. $a = -(2+\frac{1}{2}) = -5/2$, $b = 2(\frac{1}{2}) = 1$.

∴ the differentiation eq is:

Q13

$$x^2y'' + (-5/2+1)xy' + y = 0$$

$$\Rightarrow x^2y'' - 3/2\, xy' + y = 0.$$

$$W = \begin{vmatrix} x^2 & x^{1/2} \\ 2x & \frac{1}{2}x^{-1/2} \end{vmatrix} = \frac{1}{2}x^{2-1/2} - 2x^{1+1/2}$$

$$= \frac{1}{2}x^{3/2} - 2x^{3/2}$$

$$= -\frac{3}{2}x^{3/2}$$

(14) $1, e^{-2x}$, here $\lambda_1 = 0$ and $\lambda_2 = -2$.

$a = -(0-2) = +2$, $b = (0)(-2) = 0$.

∴ the diff Eq is:

$$y'' + 2y' = 0$$

Check $m^2 + 2m = 0 \Rightarrow m = \cancel{\pm\sqrt{2}} \Rightarrow m = 0, m = -2$

$\therefore y = e^{0}\cancel{(A\cos\sqrt{2}x + B\sin\sqrt{2}x)}$

$= y = c_1 e^{0x} + c_2 e^{-2x}$

(15) $\cos 2\pi x$, $\sin 2\pi x$.

Since they are L.I. So the G.S is

$$y = c_1\cos 2\pi x + c_2\sin 2\pi x.$$

$$\Rightarrow y' = -2\pi c_1\sin 2\pi x + 2\pi c_2\cos 2\pi x$$

$$\Rightarrow y'' = -4\pi^2 c_1\cos 2\pi x - 4\pi^2 c_2\sin 2\pi x$$

$$\Rightarrow y'' = -4\pi^2(c_1\cos 2\pi x + c_2\sin 2\pi x)$$

$$\Rightarrow y'' = -4\pi^2 y$$

$$\Rightarrow y'' + 4\pi^2 y = 0$$

Check $\lambda^2 + 4\pi^2 = 0 \Rightarrow \lambda = \pm 2\pi i$.

$\therefore y = (A\cos 2\pi x + B\sin 2\pi x)$

(16) $\cos(\ln x), \sin(\ln x)$

Since they are L.I.

$y = c_1\cos(\ln x) + c_2\sin(\ln x)$

$\Rightarrow y' = -\frac{1}{x}c_1\sin(\ln x) + \frac{1}{x}c_2\cos(\ln x)$

$\Rightarrow y'' = -\frac{1}{x^2}c_1\cos(\ln x) - \frac{1}{x^2}c_2\sin(\ln x)$

$\Rightarrow y'' = -\frac{1}{x^2}(c_1\cos(\ln x) + c_2\sin(\ln x))$

$\Rightarrow y'' = -\frac{1}{x^2}y \Rightarrow x^2y'' + y = 0.$

Check

$\Rightarrow m^2 + 1 = 0 \Rightarrow m = \pm i.$

$\Rightarrow$ Hence $y = [A\cos\ln x + B\sin\ln x)$

(17) $x^{3/2}, x^{-3/2}.$

here $m_1 = 3/2$ & $m_2 = -3/2$.

$\therefore a = -(3/2 - 3/2) = 0.$

$b = (3/2)(-3/2) = -9/4.$

hence the differential equation is.

$x^2y'' + (a+1)xy + by = 0$

$\Rightarrow x^2y'' + xy + 9/4y = 0$

$$W = \begin{vmatrix} x^{3/2} & x^{-3/2} \\ 3/2\,x^{1/2} & -3/2\,x^{-5/2} \end{vmatrix} = -\frac{3}{2}x^{\frac{3}{2}-\frac{5}{2}} - \frac{3}{2}x^{\frac{1}{2}-\frac{3}{2}}$$

$$= -\frac{3}{2}x^{-1} - \frac{3}{2}x^{-1}$$

$$= -3x^{-1}$$

$\therefore W = -3/x.$