

DIFFERENTIAL EQUATIONS

EXERCISE 2.14,2.15

Problems solved by;

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① General Sol.
 $y^{iv} - 16y = 0$

$$\Rightarrow \lambda^4 - 16 = 0 \Rightarrow \lambda^2 = \pm 4 \Rightarrow \lambda = \pm 2, \pm 2i$$

$$\therefore y = c_1 e^{2x} + c_2 e^{-2x} + A \cos 2x + B \sin 2x$$

Ans

② $y''' + 9y'' + 27y' + 27 = 0$

$$\Rightarrow \lambda^3 + 9\lambda^2 + 27\lambda + 27 = 0$$

It's one root is

$$(\lambda)^3 + 3(\lambda)^2(3) + 3(\lambda)(3)^2 + (3)^3 = 0$$

$$(\lambda + 3)^3 = 0$$

$$\Rightarrow \lambda = -3, -3, -3$$

$$\therefore y = c_1 e^{-3x} + c_2 x e^{-3x} + c_3 x^2 e^{-3x}$$

Ans

③ $y^{iv} - 2y'' + y = 0$

$$\lambda^4 - 2\lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^4 - \lambda^2 - \lambda^2 + 1 = 0 \Rightarrow \lambda^2(\lambda^2 - 1) - 1(\lambda^2 - 1) = 0$$

$$\Rightarrow (\lambda^2 - 1)(\lambda^2 - 1) = 0 \Rightarrow \lambda = \pm 1, \pm i$$

$$\therefore y = c_1 e^x + c_2 x e^x + c_3 e^{-x} + c_4 x e^{-x}$$

④ $y^{iv} + 2y'' + y = 0 \Rightarrow \lambda^4 + 2\lambda^2 + 1 = 0$ — ①

Let $\lambda^2 = u$. $\therefore \lambda^4 = u^2$

So ① $\Rightarrow u^2 + 2u + 1 = 0$

$$\Rightarrow u = \frac{-2 \pm \sqrt{4 - 4}}{2} = \frac{-2}{2} = -1$$

$$\Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i \text{ \& } \lambda^2 = -1 \Rightarrow \lambda = \pm i$$

$$\therefore y = A_1 \cos x + B_1 \sin x + A_2 x \cos x + B_2 x \sin x$$

$$\Rightarrow y = (A_1 + A_2 x) \cos x + (B_1 + B_2 x) \sin x$$

⑤ $y''' - 2y'' - y' + 2y = 0$

$$\Rightarrow \lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

$$\Rightarrow \lambda^2(\lambda - 2) - 1(\lambda - 2) = 0$$

$$\Rightarrow (\lambda^2 - 1)(\lambda - 2) = 0$$

$$\Rightarrow \lambda = \pm 1, 2$$

$$\textcircled{6} \quad y'''' + 5y'' + 4y = 0.$$

$$\Rightarrow \lambda^4 + 5\lambda^2 + 4 = 0$$

$$\Rightarrow \lambda^2 = \frac{-5 \pm \sqrt{25-16}}{2} = \frac{-5 \pm 3}{2} = -1, -4$$

$$\Rightarrow \lambda^2 = -1, \lambda^2 = -4.$$

$$\Rightarrow \lambda = \pm i, \lambda = \pm 2i.$$

$$\therefore y = A \cos x + B \sin x + C \cos 2x + D \sin 2x$$

$$\textcircled{7} \quad (D^3 - D^2 - D + 1)y = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 - \lambda + 1 = 0.$$

$$\Rightarrow \lambda^2(\lambda - 1) - 1(\lambda - 1) = 0 \Rightarrow (\lambda^2 - 1)(\lambda - 1) = 0.$$

$$\Rightarrow \lambda = \pm 1, \lambda = 1.$$

$$\therefore y = (C_1 + C_2 x)e^x + C_3 e^{-x}. \quad \text{Ans}$$

$$\textcircled{8} \quad (D^3 - 3D + 2)y = 0$$

$$\Rightarrow \lambda^3 - 3\lambda + 2 = 0. \quad \text{--- (1)}$$

Now by imagination. $\lambda = 1$ satisfies (1)

$$\text{ie } 1 - 3(1) + 2 = 0$$

$$\therefore \begin{array}{r} \lambda^3 - 3\lambda + 2 \\ \lambda - 1 \overline{) \lambda^3 + \lambda - 2} \\ \underline{\lambda^3 - \lambda^2} \\ \lambda^2 + \lambda - 2 \\ \underline{\lambda^2 + \lambda} \\ -2 \end{array} \quad \begin{array}{l} \Rightarrow \lambda^2 + \lambda - 2 = 0 \\ \Rightarrow \lambda^2 + 2\lambda - \lambda - 2 = 0 \\ \Rightarrow \lambda(\lambda + 2) - 1(\lambda + 2) = 0 \\ \Rightarrow \lambda = 1, \lambda = -2. \end{array}$$

$$\therefore \lambda = 1, 1, -2.$$

$$\text{so } y = (C_1 + C_2 x)e^x + C_3 e^{-2x}. \quad \text{Ans}$$

$$\textcircled{9} \quad (16D^4 - 40D^2 + 9)y = 0$$

$$\Rightarrow 16\lambda^4 - 40\lambda^2 + 9 = 0.$$

$$\Rightarrow 16\lambda^4 - 4\lambda^2 - 36\lambda^2 + 9 = 0$$

$$\Rightarrow 4\lambda^2(4\lambda^2 - 1) - 9(4\lambda^2 - 1) = 0$$

$$\Rightarrow (4\lambda^2 - 1)(4\lambda^2 - 9) = 0 \Rightarrow \lambda = \pm \frac{1}{2}, \pm \frac{3}{2}$$

$$\Rightarrow y = c_1 e^{x/2} + c_2 e^{-x/2} + c_3 e^{3x/2} + c_4 e^{-3x/2}$$

$$(10) (D^3 + 6D^2 + 11D + 6)y = 0 \quad \text{--- (1)}$$

$$\Rightarrow \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0 \quad \text{--- (2)}$$

By imagination. $\lambda = -1$ satisfies (2)

ie

$$-1 + 6 - 11 + 6 = 0.$$

\therefore we divide (2) by $\lambda + 1$ to get other roots as

$$\begin{array}{r} \lambda^2 + 5\lambda + 6 \\ \lambda + 1 \overline{) \lambda^3 + 6\lambda^2 + 11\lambda + 6} \\ \underline{\lambda^3 + \lambda^2} \\ 5\lambda^2 + 11\lambda + 6 \\ \underline{5\lambda^2 + 5\lambda} \\ 6\lambda + 6 \\ \underline{6\lambda + 6} \\ 0 \end{array} \Rightarrow \lambda^2 + 5\lambda + 6 = 0$$

$$\Rightarrow \lambda^2 + 3\lambda + 2\lambda + 6 = 0$$

$$\Rightarrow \lambda(\lambda + 3) + 2(\lambda + 3) = 0$$

$$\Rightarrow \lambda = -3, -2$$

$$\therefore \lambda = -1, -2, -3$$

and $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$ Ans

Initial Value Problems

$$(11) y^{(4)} = 0, y(0) = 1, y'(0) = 16, y''(0) = -4, y'''(0) = 24$$

$$y^{(4)} = 0$$

integrating w.r.t x

$$\Rightarrow y''' = c_1 \Rightarrow y'' = c_1 x + c_2 \Rightarrow y' = c_1 \frac{x^2}{2} + c_2 x + c_3$$

$$\Rightarrow y = \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4$$

applying initial conditions

$$\Rightarrow c_4 = 1, c_2 = -4, 16 = c_3, c_1 = 24$$

$$\therefore y = 4x^3 - 2x^2 + 16x + 1$$

$$(12) y''' - 3y'' + 3y' - y = 0, y(0) = 2, y'(0) = 2, y''(0) = 0$$

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$\Rightarrow (\lambda - 1)^3 = 0$$

$$\Rightarrow \lambda = 1, 1, 1$$

$\Rightarrow \lambda^3 - \lambda^2 - \lambda + 1 = 0$
 $\Rightarrow \lambda(\lambda^2 - 1) - 1(\lambda^2 - 1) = 0 \Rightarrow (\lambda - 1)(\lambda^2 - 1) = 0$
 $\Rightarrow \lambda = 1, 1, -1$
 $\Rightarrow y = (c_1 + c_2 x)e^x + c_3 e^{-x} \quad \text{--- (1)}$
 $\Rightarrow y' = (c_1 + c_2 x)e^x + e^x c_2 - c_3 e^{-x} \quad \text{--- (2)}$
 $\Rightarrow y'' = e^x(c_1 + c_2 x) + 2e^x c_2 + c_3 e^{-x} \quad \text{--- (3)}$
 Applying initial conditions.
 $(3) \Rightarrow 0 = c_1 + 2c_2 + c_3 \quad \text{--- (4)}$
 $(2) \Rightarrow 1 = c_1 + c_2 - c_3 \quad \text{--- (5)}$
 $(3) \Rightarrow 2 = c_1 + c_3 \quad \text{--- (6)}$
 Adding (4) and (5) $\Rightarrow 1 = 2c_1 + 3c_2 \quad \text{--- (7)}$
 Adding (5) and (6) $\Rightarrow 3 = 2c_1 + c_2 \quad \text{--- (8)}$
 Subtracting (7) and (8) $\Rightarrow 2 = -2c_2 \Rightarrow c_2 = -1$
 $\Rightarrow 2c_1 + 1 = 3 \Rightarrow c_1 = 1$
 $\Rightarrow c_3 = 2 - 1 = 1$
 So $y = (1 - x)e^x + e^{-x}$ Ans

$(14) (D^4 - 1)y = 0, \quad y(0) = 1, \quad y'(0) = 7, \quad y''(0) = -1, \quad y'''(0) = 7$
 $\lambda^4 - 1 = 0 \Rightarrow \lambda^2 = \pm 1 \Rightarrow \lambda = \pm 1, \pm i$
 $\Rightarrow y = c_1 e^x + c_2 e^{-x} + A \cos x + B \sin x \quad \text{--- (1)}$
 $\Rightarrow y' = c_1 e^x - c_2 e^{-x} - A \sin x + B \cos x \quad \text{--- (2)}$
 $\Rightarrow y'' = c_1 e^x + c_2 e^{-x} - A \cos x - B \sin x \quad \text{--- (3)}$
 $\Rightarrow y''' = c_1 e^x - c_2 e^{-x} + A \sin x - B \cos x \quad \text{--- (4)}$
 Applying initial conditions.
 $(4) \Rightarrow 7 = c_1 - c_2 - B \quad \text{--- (5)}$
 $(3) \Rightarrow -1 = c_1 + c_2 - A \quad \text{--- (6)}$
 $(2) \Rightarrow 7 = c_1 - c_2 + B \quad \text{--- (7)}$
 $(1) \Rightarrow 1 = c_1 + c_2 + A \quad \text{--- (8)}$
 Adding (5) and (7) $\Rightarrow 14 = 2c_1 \Rightarrow c_1 = 7$
 Adding (6) and (8) $\Rightarrow 0 = 2c_2 \Rightarrow c_2 = 0$
 $\Rightarrow 1 = 7 + 0 + A \Rightarrow A = -6$
 $\Rightarrow 7 = 7 - 0 - B \Rightarrow B = 0$
 So $y = 7e^x - 6 \cos x$

$$\Rightarrow 4 = 4C_1 \Rightarrow C_1 = 1.$$

$$\text{and } 2C_2 = -4 \Rightarrow C_2 = -2.$$

(9+10)

$$\text{and so } B = +4.$$

$$\text{and also } A = 0.$$

$$\therefore y = e^x - 2e^{-x} - 4\sin x. \text{ Ans}$$

$$(15) (D^3 + 5D^2 - D - 5)y = 0, \quad y(0) = 5, \quad y'(0) = 0, \quad y''(0) = 12.$$

$$\lambda^3 + 5\lambda^2 - \lambda - 5 = 0$$

$$\Rightarrow \lambda(\lambda^2 - 1) + 5(\lambda^2 - 1) = 0.$$

$$\Rightarrow (\lambda + 5)(\lambda^2 - 1) = 0$$

$$\Rightarrow \lambda = -5, -1, +1.$$

$$\therefore y = C_1 e^{-5x} + C_2 e^{-x} + C_3 e^x.$$

$$\Rightarrow y' = -5C_1 e^{-5x} - C_2 e^{-x} + C_3 e^x.$$

$$\Rightarrow y'' = 25C_1 e^{-5x} + C_2 e^{-x} + C_3 e^x$$

$$\Rightarrow y''' = -125C_1 e^{-5x} - C_2 e^{-x} + C_3 e^x.$$

Applying initial conditions

~~$$125 = -125C_1 - C_2 + C_3$$~~

$$125 = 25C_1 + C_2 + C_3. \quad \text{--- (1)}$$

$$0 = -5C_1 - C_2 + C_3. \quad \text{--- (2)}$$

$$5 = C_1 + C_2 + C_3. \quad \text{--- (3)}$$

adding (1) and (2)

$$\Rightarrow 125 = 20C_1 + 2C_3. \quad \text{--- (4)}$$

adding (2) and (3)

$$\Rightarrow 5 = -4C_1 + 2C_3. \quad \text{--- (5)}$$

subtracting (5) from (4)

$$\Rightarrow 120 = 24C_1 \Rightarrow C_1 = \frac{120}{24} = \frac{60}{12} = \frac{30}{6} = 5$$

$$\therefore C_3 = +25/2.$$

$$\therefore C_2 = \frac{-25}{2} = -\frac{25}{2}$$

$$\therefore y = 5e^{-5x} - \frac{25}{2}e^{-x} + \frac{25}{2}e^x$$

$$x^4 + 10x^2 + 9$$

$$y'''(0) = 0$$

$$\Rightarrow \lambda^4 + 9\lambda^2 + \lambda^2 + 9 = 0 \Rightarrow \lambda^2(\lambda^2 + 9) + 1(\lambda^2 + 9) = 0$$

$$\Rightarrow (\lambda^2 + 1)(\lambda^2 + 9) = 0$$

$$\Rightarrow \lambda = \pm 3i, \pm i.$$

$$\therefore y = A \cos 3x + B \sin 3x + C \cos x + D \sin x.$$

$$\text{Q16} \Rightarrow y' = -3A \sin 3x + 3B \cos 3x - C \sin x + D \cos x$$

$$\Rightarrow y'' = -9A \cos 3x - 9B \sin 3x - C \cos x - D \sin x$$

$$\Rightarrow y''' = 27A \sin 3x - 27B \cos 3x + C \sin x - D \cos x$$

Applying initial conditions

$$0 = A + C \quad \text{--- (1)}$$

$$32 = -9A - C \quad \text{--- (3)}$$

$$0 = 3B + D \quad \text{--- (2)}$$

$$0 = -27B - D \quad \text{--- (4)}$$

adding (1) and (3)

$$32 = -8A \Rightarrow A = -4 \text{ and } C = +4.$$

adding (2) and (4)

$$\Rightarrow 0 = 24B \Rightarrow B = 0 \text{ and } D = 0.$$

$$\therefore y = -4 \cos 3x + 4 \cos x. \quad (\text{Ans})$$

$$\text{(17)} \quad (D^4 + 4D^3 + 8D^2 + 8D + 4)y = 0$$

$$y(0) = 1, y'(0) = 0, y''(0) = -2, y'''(0) = 2$$

so we have

$$\lambda^4 + 4\lambda^3 + 8\lambda^2 + 8\lambda + 4 = 0.$$

$$\lambda^4 + 8\lambda^2 + 4 + 4\lambda^3 + 8\lambda = 0.$$

$$\Rightarrow \lambda^4 +$$

$$\lambda^4 + 4\lambda^3 + 8\lambda^2 + 8\lambda + 4$$

$$= \lambda^4 + 8\lambda^2 + 4 + 4\lambda^3 + 8\lambda$$

$$= \lambda^4 + 4\lambda^2 + 4 + 4\lambda^3 + 4\lambda^2 + 8\lambda$$

$$= (1+2)^2 + 4\lambda(\lambda^2 + \lambda + 2)$$

$$\lambda^4 + 4\lambda^3 + 8\lambda^2 + 8\lambda + 4$$

$$= \lambda^4 + 4\lambda^3 + 4\lambda^2 + 4\lambda^2 + 8\lambda + 4$$

$$= \lambda(\lambda^3 + 4\lambda^2 + 4\lambda + 4) + 4(\lambda^2 + 2\lambda + 1)$$

$$= \lambda(\lambda+2)^2 + 4(\lambda+1)^2$$

$$(x-a)(x-b)(x-c)(x-d) = 0$$

$$(x^2 - (a+b)x + ab)(x^2 - (c+d)x + cd) = 0$$

$$x^4 - (a+b+c+d)x^3 + (ad+bc+ab+cd)x^2$$

Ex : 2.15

(13) $y''' - 4y' = 10\cos x + 5\sin x$, $y(0) = 3$, $y'(0) = -2$
 $y''(0) = -1$.

for y_h $\lambda^3 - 4\lambda = 0 \Rightarrow \lambda(\lambda^2 - 4) = 0$.

$\Rightarrow \lambda = 0, +2, -2$.

$\therefore y_h = C_1 e^0 + C_2 e^{2x} + C_3 e^{-2x}$.

Also

$$W = \begin{vmatrix} 1 & e^{2x} & e^{-2x} \\ 0 & 2e^{2x} & -2e^{-2x} \\ 0 & 4e^{2x} & 4e^{-2x} \end{vmatrix}$$

$\Rightarrow W = 8e^{-2x} + 8e^{-2x} = 16e^{-2x}$.

$$W_1 = \begin{vmatrix} 0 & e^{2x} & e^{-2x} \\ 0 & 2e^{2x} & -2e^{-2x} \\ 1 & 4e^{2x} & 4e^{-2x} \end{vmatrix} = -4$$

$$W_2 = \begin{vmatrix} 1 & 0 & e^{-2x} \\ 0 & 0 & -2e^{-2x} \\ 0 & 1 & 4e^{-2x} \end{vmatrix} = +2e^{-2x}$$

$$W_3 = \begin{vmatrix} 1 & e^{2x} & 0 \\ 0 & 2e^{2x} & 0 \\ 0 & 4e^{2x} & 1 \end{vmatrix} = 2e^{2x}$$

$\therefore y_p = y_1 \int \frac{W_1}{W} r(x) dx + y_2 \int \frac{W_2}{W} r(x) dx + y_3 \int \frac{W_3}{W} r(x) dx$

$\Rightarrow y_p = \int \frac{-4}{16e^{-2x}} (10\cos x + 5\sin x) dx + \int \frac{2e^{-2x}}{16e^{-2x}} (10\cos x + 5\sin x) dx$
 $+ e^{-2x} \int \frac{2e^{2x}}{16e^{2x}} (10\cos x + 5\sin x) dx$

$$= \frac{-1}{4} \int e^{2x} (10 \cos x + 5 \sin x) dx$$

$$x^3 D^3 y + x D y - y = x^2$$

$$\therefore x^3 D^3 (\Delta(\Delta-1)(\Delta-2) + \Delta - 1) y = e^{2t}$$

$$\therefore (\Delta^3 - 3\Delta^2 + 2\Delta + \Delta - 1) y = e^{2t}$$

$$\therefore (\Delta^3 - 3\Delta^2 + 3\Delta - 1) y = e^{2t}$$

$$y = A \cos x + B \sin x, \quad y' = -A \sin x + B \cos x$$

$$y'' = -A \cos x - B \sin x, \quad y''' = A \sin x - B \cos x$$

$$y''' - 4y' = 10 \cos x + 5 \sin x$$

$$A \sin x - B \cos x - 4(-A \sin x + B \cos x) = 10 \cos x + 5 \sin x$$

$$\text{or } 5A \sin x - 5B \cos x = 5 \sin x + 10 \cos x$$

$$A = 1, \quad B = -2$$

$$y_p = \cos x - 2 \sin x$$

$$y = y_h + y_p = C_1 + C_2 e^{2x} + C_3 e^{-2x} + \cos x - 2 \sin x$$

$$y(0) = C_1 + C_2 + C_3 + 1 = -1$$

$$y' = 2C_2 - 2C_3 = 0 \quad C_2 = C_3$$

$$y'' = 4C_2 + 4C_3 = 0 \quad C_2 = -C_3 \quad C_2 = C_3 = 0$$

(10)

$$x^3 y''' + xy' - y = x^2, \quad y(1) = 1, \quad y'(1) = 3, \quad y''(1) = 3$$

$$x = e^t \Rightarrow t = \ln x, \quad \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt}$$

$$\text{So } x \frac{dy}{dx} = \frac{dy}{dt} \quad \text{or } x D_y = D_t$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{1}{x} \frac{dy}{dt} \right) \cdot \frac{dt}{dx} = \frac{1}{x} \left(\frac{1}{x} \frac{d^2 y}{dt^2} - \frac{1}{x^2} \frac{dy}{dt} \right)$$

$$\text{Let } x = e^t \Rightarrow t = \ln x$$

$$\frac{dx}{dt} = e^t = x \quad \text{and} \quad \frac{dt}{dx} = \frac{1}{x}$$

$$\text{Now } D_y = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x} = \frac{1}{x} \frac{dy}{dt} = \frac{1}{x} D_t$$

$$\text{So, } D_y = \frac{1}{x} D_t \quad \text{or} \quad D_t = x D_y \quad \text{--- (1)}$$

$$\text{Now } \frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx} = \frac{d}{dt} \left(\frac{1}{x} \frac{dy}{dt} \right) \cdot \frac{1}{x}$$

$$\begin{aligned}
 &= \left(\frac{1}{x} \frac{dy}{dx^2} + \frac{d}{dx} \left(\frac{1}{x} \right) \cdot \frac{dy}{dx} \right) \cdot \frac{1}{x} = \frac{1}{x} \left(\frac{1}{x} \frac{dy}{dx^2} - \frac{1}{x} \frac{dy}{dx} \right) \\
 &= \frac{1}{x^2} \left(\frac{dy}{dx^2} - \frac{dy}{dx} \right) = \frac{1}{x^2} \frac{d}{dx} \left(\frac{dy}{dx} - y \right) \\
 &= \frac{1}{x^2} (\Delta^2 y - \Delta y) = \frac{1}{x^2} (\Delta^2 - \Delta) y
 \end{aligned}$$

$$\text{So, } D^2 y = \frac{1}{x^2} (\Delta^2 - \Delta) y$$

$$\text{or } x^2 D^2 y = (\Delta^2 - \Delta) y = \Delta(\Delta - 1) y \quad (2)$$

$$\text{Now } D^3 y = \frac{d^3 y}{dx^3} = \frac{d}{dx} \left(\frac{dy}{dx^2} \right) = \frac{d}{dx} \left(\frac{dy}{dx} \right) \cdot \frac{dx}{dx}$$

$$= \frac{d}{dx} \left(\frac{dy}{dx^2} \right) \cdot \frac{1}{x} = \frac{1}{x} \frac{d}{dx} \left(\frac{1}{x} \left(\frac{dy}{dx^2} - \frac{dy}{dx} \right) \right)$$

$$= \frac{1}{x} \left(\frac{1}{x^2} \left(\frac{d^3 y}{dx^3} - \frac{dy}{dx^2} \right) - \frac{2}{x^3} \frac{d}{dx} \left(\frac{dy}{dx} - y \right) \right)$$

$$= \frac{1}{x} \left(\frac{1}{x^2} \left(\frac{d^3 y}{dx^3} - \frac{dy}{dx^2} \right) - \frac{2}{x^3} \left(\frac{dy}{dx^2} - \frac{dy}{dx} \right) \right)$$

$$\left[\frac{d}{dx}(x) = \frac{d}{dx}(e^t) = e^t = x \right]$$

$$= \frac{1}{x^3} \left(\frac{d^3 y}{dx^3} - \frac{dy}{dx^2} - 2 \left(\frac{dy}{dx^2} - \frac{dy}{dx} \right) \right)$$

$$= \frac{1}{x^3} \left(\frac{d^3 y}{dx^3} - 3 \frac{dy}{dx^2} + 2 \frac{dy}{dx} \right) = \frac{1}{x^3} (\Delta^3 - 3\Delta^2 + 2\Delta) y$$

$$\text{Thus } D^3 y = \frac{1}{x^3} (\Delta^3 - 3\Delta^2 + 2\Delta) y$$

$$\text{or } x^3 D^3 y = (\Delta^3 - 3\Delta^2 + 2\Delta) y$$

$$= \Delta(\Delta^2 - 3\Delta + 2) y$$

$$x^3 D^3 y = \Delta(\Delta - 1)(\Delta - 2) y \quad (3)$$

Thus, we have $x D = \Delta$

$$x^2 D^2 = \Delta(\Delta - 1), \quad x^3 D^3 = \Delta(\Delta - 1)(\Delta - 2),$$

$$x^4 D^4 = \Delta(\Delta - 1)(\Delta - 2)(\Delta - 3) \text{ and so on}$$

$$\text{upto } x^n D^n = \Delta(\Delta - 1)(\Delta - 2)(\Delta - 3) \dots (\Delta - (n-1))$$

where $D = \frac{d}{dx}$, $\Delta = \frac{d}{dx}$ and $n \in \mathbb{Z}^+$.

$$1 - 6 + 11 - 6 = 0$$

$$(11) \quad x^3 y''' - 3x^2 y'' + 6xy' - 6y = 24x^5$$

$$y(1)=1, y'(1)=3, y''(1)=14.$$

for y_h .

$$m(m-1)(m-2) - 3m(m-1) + 6m - 6 = 0$$

$$\Rightarrow (m^2 - m)(m-2) - 3(m^2 - m) + 6m - 6 = 0.$$

$$Q1 \Rightarrow m^3 - 3m^2 + 2m - 3m^2 + 3m + 6m - 6 = 0$$

$$\Rightarrow m^3 - 6m^2 + 11m - 6 = 0.$$

Its one root is $m=1$.

because for $m=1$.

$$1 - 6 + 11 - 6 = 12 - 12 = 0.$$

other roots can be obtained by division by $m-1$

$$\begin{array}{r} m^3 - 6m^2 + 11m - 6 \\ m-1 \overline{) m^3 - 6m^2 + 11m - 6} \\ \underline{m^3 - m^2} \\ -5m^2 + 11m \\ \underline{-5m^2 + 5m} \\ 6m - 6 \\ \underline{6m - 6} \\ 0 \end{array}$$

ie $m-1 \sqrt{m^3 - 6m^2 + 11m - 6}$

we have $m^2 - 5m + 6 = 0$

$$\Rightarrow m^2 - 3m - 2m + 6 = 0$$

$$\Rightarrow (m-3)(m-2) = 0$$

$$\Rightarrow m=2, m=3$$

$$\therefore y_h = C_1 x + C_2 x^2 + C_3 x^3$$

Now

$$W = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \end{vmatrix}$$

$$\Rightarrow W = \begin{vmatrix} 0 & x^2 & 2x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} \xrightarrow{+xR_2 - R_1} \begin{vmatrix} 0 & x^2 & 2x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} \xrightarrow{+xR_2 - R_1} \begin{vmatrix} 0 & x^2 & 2x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$\therefore W = 6x^3 - 4x^3 + 2x^3$$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4$$

$$\begin{array}{r} x^2 \quad x^2 \\ x \quad -x \quad 2x \\ 2 \quad -0 \quad 2 \\ \hline x \cdot 2x^2 \quad 3x^3 \\ \underline{-x^2 \quad x^3} \\ 0 \quad x^2 \quad 2x^3 \\ \hline x \quad x^2 \quad x^3 \\ 0 \quad x^2 \quad 5x^3 \\ \hline 0 \quad 2 \quad 6x \\ x(6x^3 - 10x^3) \\ \Rightarrow x(-4x^3) \\ = -4x^4 \end{array}$$

$$W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2.$$

$$y_p = x \int \frac{x^4}{2x^3} \cdot 24x^2 dx + x^2 \int \frac{2x^3}{2x^3} \cdot 24x^2 dx + x^3 \int \frac{x^2}{2x^3} \cdot 24x^2 dx.$$

$$\Rightarrow y_p = 12x \int x^3 dx + 24x^2 \int x^2 dx + 12x^3 \int x dx.$$

$$\Rightarrow y_p = 12x \left(\frac{x^4}{4} \right) + 24x^2 \left(\frac{x^3}{3} \right) + 12x^3 \left(\frac{x^2}{2} \right)$$

$$\Rightarrow y_p = 3x^5 + 8x^5 + 6x^5 = 17x^5.$$

$$\therefore y = C_1 x + C_2 x^2 + C_3 x^3 + 17x^5.$$

$$\Rightarrow y' = C_1 + 2C_2 x + 3C_3 x^2 + 85x^4.$$

$$\Rightarrow y'' = 2C_2 + 6C_3 x + 340x^3$$

$$\Rightarrow y''' = 6C_3 + 1020x^2$$

applying initial conditions.

$$1 = C_1 + C_2 + C_3 + 17 \Rightarrow C_1 + C_2 + C_3 = -16. \quad \text{--- (1)}$$

$$3 = C_1 + 2C_2 + 3C_3 + 85 \Rightarrow C_1 + 2C_2 + 3C_3 = -82. \quad \text{--- (2)}$$

$$14 = 2C_2 + 6C_3 + 340 \Rightarrow 2C_2 + 6C_3 = -326. \quad \text{--- (3)}$$

$$\text{subtracting (1) from (2)} \Rightarrow C_2 + 3C_3 = -66. \quad \text{--- (4)}$$

$$\Rightarrow C_2 + 2C_3 = -66. \quad \text{--- (4)}$$

$$\text{subtracting (4) from (3)}$$

$$\Rightarrow C_3 = -97$$

$$\text{Hence } C_2 = 128.$$

$$\& \quad C_1 = -47.$$

$$(9) \quad y''' - 3y'' + 3y' + y = e^x \sin x.$$

$$\lambda^3 - 3\lambda^2 + 3\lambda + 1 = 0$$

$$\Rightarrow \lambda^2(\lambda - 3) -$$

$$99 \quad \lambda(\lambda^2 + 3) - 3(\lambda^2 + 1) = 0 \quad \lambda = +1, -1, -1.$$

$$y_h = (C_1 + C_2 x + C_3 x^2) e^{-x}.$$

$$y_p = e^x (A \cos x + B \sin x)$$

$$y_p = \underline{\underline{A e^x \cos x.}}$$

$$y(0)=0, y'(0)=6, y''(0)=0, y'''(0)=-26$$

$$\lambda^4 + 16\lambda^2 + 9 = 0$$

$$(\lambda^2 + 9)(\lambda^2 + 1) = 0$$

$$\lambda^2 = -9, \lambda^2 = -1$$

$$\lambda = \pm 3i, \pm i$$

$$y_h = C_1 \cos 3x + C_2 \sin 3x + C_3 \cos x + C_4 \sin x$$

$$y_p = A \cosh x + B \sinh x$$

$$y_p' = A \sinh x + B \cosh x$$

$$y_p'' = A \cosh x + B \sinh x$$

$$y_p''' = A \sinh x + B \cosh x$$

$$y_p^{(4)} = A \cosh x + B \sinh x$$

$$(A \cosh x + B \sinh x) + 16(A \cosh x + B \sinh x) + 9(A \cosh x + B \sinh x) = 40 \sinh x$$

$$\Rightarrow 26A \cosh x + 26B \sinh x = 40 \sinh x$$

$$\Rightarrow A=0, B=2$$

$$y_p = 2 \sinh x$$

$$y_p' = 2 \cosh x, y_p'' = 2 \sinh x, y_p''' = 2 \cosh x, y_p^{(4)} = 2 \sinh x$$

$$2 \sinh x + 32 \sinh x + 18 \sinh x = 40 \sinh x$$

$$y = C_1 \cos 3x + C_2 \sin 3x + C_3 \cos x + C_4 \sin x + 2 \sinh x$$

$$\Rightarrow y' = -3C_1 \sin 3x + 3C_2 \cos 3x - C_3 \sin x + C_4 \cos x + 2 \cosh x$$

$$\Rightarrow y'' = -9C_1 \cos 3x - 9C_2 \sin 3x - C_3 \cos x - C_4 \sin x + 2 \sinh x$$

$$\Rightarrow y''' = 27C_1 \sin 3x - 27C_2 \cos 3x + C_3 \sin x - C_4 \cos x + 2 \cosh x$$

$$\Rightarrow y^{(4)} = 81C_1 \cos 3x + 81C_2 \sin 3x + C_3 \cos x + C_4 \sin x + 2 \sinh x$$

$$+ 28 \sin x - 40 C_1 \cos 3x - 90 C_2 \sin 3x - 10 C_3 \cos x \\ - 10 C_4 \sin x + 20 \sin x + 9 C_1 \cos 3x + 9 C_2 \sin 3x \\ + 9 C_3 \cos x + 9 C_4 \sin x + 18 \sin x.$$

$$y_p = A e^{3x} + B e^{-3x}$$

$$\Rightarrow y'_p = 3A e^{3x} - 3B e^{-3x}$$

$$\Rightarrow y''_p = 9A e^{3x} + 9B e^{-3x}$$

$$y'''_p = 27A e^{3x} - 27B e^{-3x}$$

$$y^{(4)}_p = 81A e^{3x} + 81B e^{-3x}$$

$$81A e^{3x} + 18B e^{-3x} + 90A e^{3x} + 90B e^{-3x} + 9A e^{3x} + 9B e^{-3x} \\ = 20 e^{3x}$$

$$y = A e^x + B e^{-x}$$

$$y' = A e^x - B e^{-x}$$

$$y'' = A e^x + B e^{-x}$$

$$y''' = A e^x - B e^{-x}$$

$$y^{(4)} = A e^x + B e^{-x}$$

$$A e^x + B e^{-x} + 18A e^x + 18B e^{-x} + 9A e^x + 9B e^{-x} \\ 20A e^x + 20B e^{-x} = 20 e^x + 20 e^{-x}$$

$$A = 1, B = 1$$

$$\therefore y = e^x + e^{-x} = 2 \cosh x$$

y

$$y_h = Ae^{2x} + Be^{-2x} + C \cos 2x + D \sin 2x.$$

$$\text{Q.B } y_h = M \cos x + N \sin x.$$

$$\Rightarrow y'_h = -M \sin x + N \cos x$$

$$\Rightarrow y''_h = -M \cos x - N \sin x$$

$$\Rightarrow y'''_h = M \sin x - N \cos x$$

$$\Rightarrow y^{(4)}_h = M \cos x + N \sin x.$$

\Rightarrow

$$M \cos x + N \sin x = 4$$

$$(13) \quad y^{(4)} - 4y = 10 \cos x + 5 \sin x. \quad (1)$$

$$\Rightarrow \lambda^4 - 4\lambda = 0 \Rightarrow \lambda(\lambda^3 - 4) \Rightarrow \lambda = 0, 2, -2.$$

$$\Rightarrow y_h = C_1 + C_2 e^{2x} + C_3 e^{-2x}.$$

$$y_1 = A \cos x + B \sin x.$$

$$\Rightarrow y'_1 = -A \sin x + B \cos x.$$

$$\Rightarrow y''_1 = -A \cos x - B \sin x.$$

$$\Rightarrow y'''_1 = A \sin x - B \cos x$$

$$(1) \Rightarrow A \sin x - B \cos x + 4A \sin x - 4B \cos x = 10 \cos x + 5 \sin x$$

$$\Rightarrow 5A \sin x - 5B \cos x = 10 \cos x + 5 \sin x$$

$$\Rightarrow 5A = 5 \Rightarrow A = 1, B = -2.$$

$$\therefore y_p = \sin x - 2 \cos x.$$

$$\therefore y_h = C_1 + C_2 e^{2x} + C_3 e^{-2x} + \sin x - 2 \cos x.$$

$$y'_h = 2C_2 e^{2x} - 2C_3 e^{-2x} + \cos x + 2 \sin x$$

$$y''_h = 4C_2 e^{2x} + 4C_3 e^{-2x} - \sin x + 2 \cos x$$

$$C_1 + C_2 + C_3 = 3$$

$$\Rightarrow C_1 + C_2 + C_3 = 3$$

$$2C_2 - 2C_3 = -2 \Rightarrow 2C_2 - 2C_3 = -2$$

$$\Rightarrow -2 \Rightarrow 4C_2 + 4C_3 = -1$$

$$\Rightarrow 2(2C_2 + 2C_3) = -2$$

$$4C_2 - 4C_3 = -6$$

$$4C_2 + 4C_3 = -2$$

$$A \sin x - B \cos x + 4A \sin x - 4B \cos x = 10 \cos x + 5 \sin x$$

$$\Rightarrow \underline{\cos x - 2 \sin x}$$

$$2C_2 + 2C_3 = -1$$

$$\Rightarrow 2C_2 - 2C_3 = 0$$

$$\underline{4C_2 = -1} \Rightarrow C_2 = -1/4$$

$$C_3 = -1/2 = -2C_2$$

$$\Rightarrow C_3 = 1/2$$

$$-1/4 + 1/2 + C_1 = 5$$

$$\Rightarrow C_1 = 8$$

$$\Rightarrow 2 - \frac{1}{4} e^{2x} + \frac{1}{2} e^{-2x} + \sin x - 2 \cos x$$

$$\Rightarrow \underline{y_p = 2 - \frac{1}{2} \sinh x + \sin x - 2 \cos x}$$

$$\underline{y_p = \cos x - 2 \sin x}$$

$$y = C_1 + C_2 e^{2x} + C_3 e^{-2x} + \cos x - 2 \sin x$$

$$y' = 2C_2 e^{2x} - 2C_3 e^{-2x} + \sin x - 2 \cos x$$

$$y'' = 4C_2 e^{2x} + 4C_3 e^{-2x} - \cos x + 2 \sin x$$

$$-1 = 4C_2 + 4C_3 - 1$$

$$\Rightarrow 4C_2 + 4C_3 = 0 \Rightarrow C_2 + C_3 = 0$$

$$-2 = 2C_2 - 2C_3 - 2 \Rightarrow 2C_2 - 2C_3 = 0 \Rightarrow C_2 - C_3 = 0$$

$$C_1 + C_2 + C_3 + 1 = 3 \Rightarrow \underline{C_1 = 2}$$