

DIFFERENTIAL EQUATIONS

EXERCISE 1.3

Problems solved by;

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Q#3 — Q#23

Q#3 $y' = 1 + 0.01y^2$

Solution $y' = 1 + 0.01y^2$

$$\Rightarrow \frac{dy}{dx} = 1 + 0.01y^2$$

$$\Rightarrow \frac{1}{(1 + 0.01y^2)} \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \int \frac{1}{(1 + 0.01y^2)} \cdot \frac{dy}{dx} \cdot dx = \int dx$$

 $(\because \text{integrating b/s w.r.t } x)$

$$\Rightarrow \int \frac{dy}{(1 + 0.01y^2)} = \int dx$$

$$\Rightarrow \tan^{-1}(0.1y) = x + C$$

$$\Rightarrow y = 10 \tan(x + C)$$

Ans

Q#5 $y' = xy/2$

Solution

$y' = xy/2$

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{2} \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{x}{2}$$

$$\Rightarrow \int \frac{1}{y} \cdot \frac{dy}{dx} \cdot dx = \int \frac{x}{2} dx$$

 $(\because \text{integrating b/s w.r.t } x)$

$$\Rightarrow \int \frac{dy}{y} = \frac{1}{2} \int x dx$$

$$\Rightarrow \ln y = \frac{1}{2} \cdot \frac{x^2}{2} + C$$

$$\Rightarrow y = e^{x^2/4 + C}$$

$$\Rightarrow y = ce^{x^2/4} \quad = e^c = e$$

Q#7 $xy' = y^2 + y$ $(y/x = u)$

Solution

$$xy' = y^2 + y$$

$$\Rightarrow y' = y(y/x) + (y/x) \quad \text{--- (1)}$$

Let $y/x = u \Rightarrow y = ux \Rightarrow y' = xu' + u$
 putting in (1)

$$\Rightarrow xu' + u = ux(u) + u$$

$$\Rightarrow u' = u^2$$

$$\Rightarrow \frac{du}{dx} = u^2 \Rightarrow \frac{1}{u^2} \cdot \frac{du}{dx} = 1$$

$$\Rightarrow \int \frac{1}{u^2} \cdot \frac{du}{dx} \cdot dx = \int dx$$

(\Rightarrow integrating b/s w.r.t x)

$$\Rightarrow \int \frac{du}{u^2} = \int dx$$

$$\Rightarrow -\frac{1}{u} = x + C$$

$$\Rightarrow -\frac{x}{y} = x + C \quad (\because u = y/x)$$

$$\Rightarrow -x = y(x + C)$$

$$\Rightarrow y = -\frac{x}{(x+C)}$$

Ans

Solution

$$\text{Q9} \quad y' = (x^2 + y^2) / xy$$

$$\Rightarrow y' = \frac{x}{y} + \frac{y}{x} \quad \text{--- ①}$$

$$\text{Let } y/x = u \Rightarrow y = ux \Rightarrow y' = xu' + u$$

(substituting in ①)

$$\Rightarrow y' = xu' + u = \frac{1}{u} + u$$

$$\Rightarrow xu' = \frac{1}{u}$$

$$\Rightarrow uu' = \frac{1}{x}$$

$$\Rightarrow u \frac{du}{dx} = \frac{1}{x}$$

$$\Rightarrow \int u \frac{du}{dx} \cdot dx = \int \frac{1}{x} dx$$

(\because integrating b/s wrt x)

$$\Rightarrow \int u du = \int \frac{1}{x} dx$$

($\because (\frac{du}{dx}) dx = du$)

$$\Rightarrow \frac{u^2}{2} = \ln x + \ln C_1$$

$$\Rightarrow u^2 = 2 \ln x + \ln C \quad (\because C = 2C_1)$$

$$\Rightarrow \frac{y^2}{x^2} = 2 \ln x + \ln C \quad (\because u = y/x)$$

$$\Rightarrow y^2 = 2x^2 \ln x + \ln C \cdot x^2$$

$$\Rightarrow y = x \sqrt{2 \ln x + \ln C} \quad \text{Ans}$$

Solution

$$y' + \operatorname{cosec} y = 0$$

$$\Rightarrow \frac{dy}{dx} = -\operatorname{cosec} y$$

$$\Rightarrow \frac{1}{\operatorname{cosec} y} \cdot \frac{dy}{dx} = -1$$

$$\Rightarrow \int \frac{1}{\operatorname{cosec} y} \cdot \frac{dy}{dx} \cdot dx = \int -1 \cdot dx$$

(\because integrating b/s w.r.t x)

$$\Rightarrow \int \sin y \, dy = -\int dx$$

$$\left(\because \left(\frac{dy}{dx} \right) dx = dy \right)$$

$$\Rightarrow -\cos y = -x + C$$

$$\Rightarrow y = \cos^{-1}(x+C) \quad \because -C_1 = C$$

Ans

Q # 13 $xy' + y = 0$, $y(2) = -2$

Solution

$$\Rightarrow x \frac{dy}{dx} = -y$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = -\frac{1}{x}$$

$$\Rightarrow \int \frac{1}{y} \cdot \frac{dy}{dx} \cdot dx = \int -\frac{1}{x} \cdot dx$$

(\because integrating b/s w.r.t x)

$$\Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\Rightarrow \ln y = -\ln x + \ln C$$

$$\Rightarrow y = \frac{C}{x} \quad \text{--- (1)}$$

Now $y(2) = -2$, Given.

Putting in (1)

Substituting in (1)

$$\Rightarrow y = -4/x \Rightarrow xy = -4$$

Ans

Q#15

$$e^x y' = 2(x+1)y^2, \quad y(0) = 1/6$$

Solution

$$e^x \frac{dy}{dx} = 2(x+1)y^2$$

$$\Rightarrow \frac{1}{y^2} \frac{dy}{dx} = \frac{2(x+1)}{e^x}$$

(Integrating b/s w.r.t x)

$$\Rightarrow \int \frac{1}{y^2} \frac{dy}{dx} \cdot dx = \int \frac{2(x+1)}{e^x} dx$$

$$\Rightarrow \int \frac{dy}{y^2} = 2 \int e^{-x} (x+1) dx$$

$$\Rightarrow -\frac{1}{y} = +2 \left(e^{-x} (x+1) - \int -e^{-x} dx \right) + C$$

$$\Rightarrow -\frac{1}{y} = -2 \left(e^{-x} (x+1) + (-e^{-x}) \right) + C$$

$$\Rightarrow \frac{1}{y} = 2 \left(e^{-x} (x+1) + e^{-x} \right) + C \quad \text{--- (1)}$$

$$\text{Now } y(0) = 1/6 \quad \text{putting in (1)}$$

$$\Rightarrow 6 = 2(e^0(0+1) + e^0) + C$$

$$\Rightarrow 6 = 4 + C$$

$$\Rightarrow C = 2$$

Substituting in (1)

$$\Rightarrow y = \frac{1}{e^{-2/(2x+2)} + 2} + 2$$

$$2 + (2x+4)e^x$$

Ans

Q # 17

$$y' \cosh^2 x - \sin^2 y = 0, \quad y(0) = \pi/2$$

Solution

$$\Rightarrow \frac{dy}{dx} \cosh^2 x = \sin^2 y$$

$$\Rightarrow \frac{1}{\sin^2 y} \cdot \frac{dy}{dx} = \frac{1}{\cosh^2 x}$$

$$\Rightarrow \operatorname{cosec}^2 y \cdot \frac{dy}{dx} = \operatorname{sech}^2 x$$

integrating w/s w.r.t x.

$$\Rightarrow \int \operatorname{cosec}^2 y \cdot \frac{dy}{dx} \cdot dx = \int \operatorname{sech}^2 x \, dx$$

$$\Rightarrow \int \operatorname{cosec}^2 y \, dy = \int \operatorname{sech}^2 x \, dx$$

$$\therefore \left(\frac{dy}{dx}\right) dx = dy$$

$$\Rightarrow -\cot y = \tanh x + C \quad \text{--- (1)}$$

now $y(0) = \pi/2$ putting in (1)

$$\Rightarrow -\cot \pi/2 = \tanh 0 + C$$

$$\Rightarrow 0 = C$$

$$\Rightarrow C = 0 \quad \text{substituting in (1)}$$

$$\Rightarrow -\cot y = \tanh x + 0$$

$$\Rightarrow y = \cot^{-1}(-\tanh x)$$

Ans

(L & R are constants)

Solution

$$\Rightarrow L \left(\frac{dI}{dt} \right) = -RI$$

$$\Rightarrow \frac{1}{I} \cdot \frac{dI}{dt} = -\frac{R}{L}$$

Integrating b/s w.r.t. "t"

$$\Rightarrow \int \frac{1}{I} \cdot \frac{dI}{dt} \cdot dt = -\frac{R}{L} \int dt$$

$$\Rightarrow \int \frac{1}{I} dI = -\frac{R}{L} \int dt$$

$$\Rightarrow \left(\frac{dI}{dt} \right) dt = dI$$

$$\Rightarrow \ln I = -\frac{R}{L} t + \ln C$$

$$\Rightarrow I/C = e^{-\frac{R}{L} t}$$

$$\Rightarrow I = C e^{-\frac{R}{L} t} \quad \text{--- (1)}$$

Now $I(0) = I_0$ pulling in (1)

$$\Rightarrow I_0 = C \quad \text{substituting in (1)}$$

$$\Rightarrow I = I_0 e^{-\frac{R}{L} t}$$

Ans

Solution

$$\Rightarrow y' = \frac{y}{x} + 3x^3 \cos^2(y/x) \quad \text{--- (1)}$$

Q21

$$\text{let } y/x = u \Rightarrow y = ux.$$

$$\Rightarrow y' = xu' + u \quad \text{putting in (1)}$$

$$\Rightarrow xu' + u = u + 3x^3 \cos^2 u.$$

$$\Rightarrow u' = 3x^2 \cos^2 u.$$

$$\Rightarrow \frac{du}{dx} = 3x^2 \cos^2 u.$$

$$\Rightarrow \frac{1}{\cos^2 u} \cdot du = \int 3x^2.$$

$$\Rightarrow \sec^2 u \cdot \frac{du}{dx} = 3x^2.$$

integrating b/s w.r.t x.

$$\Rightarrow \int \sec^2 u \cdot \frac{du}{dx} \cdot dx = \int 3x^2 dx.$$

$$\Rightarrow \int \sec^2 u \cdot du = 3 \int x^2 dx.$$

$$\Rightarrow \left(\frac{du}{dx}\right)(dx) = du.$$

$$\Rightarrow \tan u = \frac{3x^3}{3} + C$$

$$\Rightarrow \tan u = x^3 + C.$$

$$\Rightarrow \tan y/x = x^3 + C$$

$$\Rightarrow y = x \tan^{-1}(x^3 + C) \quad \text{--- (2)} \quad \text{where } u = y/x.$$

$$\Rightarrow 0 = \tan^{-1}(1+C).$$

$$\Rightarrow C+1 = \tan 0$$

$$\Rightarrow C = -1$$

$$\Rightarrow y = x \tan^{-1}(x^3 - 1).$$

Ans

Q#23 $xyy' = 2y^2 + 4x^2$, $y(2) = 4$

Solution

$$xyy' = 2y^2 + 4x^2$$

$$\Rightarrow y' = 2y/x + 4x/y \quad \text{--- ①}$$

Let $u = y/x \Rightarrow y = ux \Rightarrow y' = xu' + u$
 putting in ①

$$\Rightarrow xu' + u = 2u + \frac{4}{u}$$

$$\Rightarrow xu' = u + \frac{4}{u}$$

$$\Rightarrow xu' = \frac{u^2 + 4}{u}$$

$$\Rightarrow \frac{4}{u^2 + 4} \frac{du}{dx} = \frac{1}{x}$$

integrating b/s w.r.t x.

$$\Rightarrow \int \frac{4}{u^2 + 4} \frac{du}{dx} \cdot dx = \int \frac{1}{x} dx.$$

$$\Rightarrow \frac{1}{2} \int \frac{2u}{u^2 + 4} du = \int \frac{1}{x} dx.$$

$$= \left(\frac{du}{dx} \right) dx = du$$

$$\Rightarrow \ln(u^2 + 4)^{1/2} = \ln(cx)$$

$$\Rightarrow (u^2 + 4)^{1/2} = cx$$

$$\Rightarrow \left(\frac{y^2}{x^2} + 4\right)^{1/2} = cx \quad (\because u = y/x)$$

$$\Rightarrow \frac{y^2}{x^2} + 4 = c^2 x^2$$

$$\Rightarrow y^2 + 4x^2 = c^2 x^4 \quad \text{--- (2)}$$

$$\text{Now } y(2) = 4.$$

$$\Rightarrow 16 + 4(4) = c^2(16)$$

$$\Rightarrow 32 = c^2(16)$$

$$\Rightarrow c^2 = 2 \Rightarrow c = \sqrt{2} \quad \text{putting in (2)}$$

$$\Rightarrow y^2 + 4x^2 = 2x^4$$

$$\Rightarrow y^2 = 2x^4 - 4x^2$$

$$\Rightarrow y = \sqrt{2x^4 - 4x^2}$$

Ans

Many 1st order differential equations reduce to the form:

$$(1) \quad g(y)y' = f(x) \quad \text{by algebraic manipulations.}$$

$$\because y' = dy/dx$$

$$\Rightarrow g(y) \frac{dy}{dx} = f(x) \Rightarrow g(y) dy = f(x) dx \quad \text{--- (2)}$$

Such an equation is called separable equation, because in (2) the variables x and y are separated so that x appears only on the right and y on the left.

To solve (1), integrate both sides w.r.t x obtaining

$$\int g(y) \frac{dy}{dx} dx = \int f(x) dx + c.$$

Now on the left we can switch to y as the variable of integration.

By calculus $(dy/dx)(dx) = dy$, so that we get

$$\int g(y) dy = \int f(x) dx + c \quad \text{--- (3)}$$

If we assume that f and g are continuous functions, the integrals in (3) will exist, and by evaluating these integrals we obtain the general solution of (1).

REDUCTION TO SEPERABLE FORM

Certain differential equations are not separable but can be made separable by the introduction of new unknown function, as in the following example.

EXAMPLE 6 (Book)

Solve $2xyy' = y^2 - x^2$

$$\Rightarrow y' = \frac{y^2}{2xy} - \frac{x^2}{2xy} = \frac{1}{2} \left(\frac{y}{x} - \frac{x}{y} \right) \quad \text{--- (1)}$$

putting $y/x = u \Rightarrow y = ux \Rightarrow y' = xu' + u$.

$\Rightarrow y' = xu' + u = \frac{1}{2}(u + \frac{1}{u})$. putting in ①

$\Rightarrow u'x = \frac{1}{2}u - \frac{1}{2u} - u = -\frac{1}{2u} - \frac{1}{2}u$.

$\Rightarrow u'x = -\frac{1}{2}(\frac{1}{u} + u) = -\frac{1}{2}(\frac{1+u^2}{u})$.

$\Rightarrow u'x = -\frac{1}{2}(\frac{1+u^2}{u})$.

$\Rightarrow \int \frac{2u}{1+u^2} du = \int -\frac{dx}{x}$

$\Rightarrow \ln(1+u^2) = -\ln x + \ln c$.

$\Rightarrow 1+u^2 = c/x$

Now putting $u = y/x$.

$\Rightarrow 1 + \frac{y^2}{x^2} = c/x \Rightarrow x^2 + y^2 = cx$

$\Rightarrow x^2 - cx + \frac{c^2}{4} - \frac{c^2}{4} + y^2 = 0$

$\Rightarrow (x - \frac{c}{2})^2 + y^2 = \frac{c^2}{4}$

Ans

Transformations.

Sometimes a transformation $v = ay + bx + k$ may lead to a separable differential equation as in the following example.

Example 7 (book)

$(2x - 4y + 5)y' + x - 2y + 3 = 0$. ①

put $x - 2y = u \Rightarrow y = \frac{x-u}{2} \Rightarrow y' = \frac{1-u'}{2}$

$\Rightarrow (2u + 5)(\frac{1-u'}{2}) + u + 3 = 0$ putting in ①

$\Rightarrow (2u + 5)(1 - u') = -2u - 6$

$$\Rightarrow (4u+10)u' = 8u+22.$$

$$\Rightarrow \frac{4u+10}{4u+11} du = 2dx \Rightarrow \frac{4u+11-1}{4u+11} du = 2dx$$

$$\Rightarrow \int du - \int \frac{1}{4u+11} du = \int 2dx.$$

$$\Rightarrow u - \frac{1}{4} \ln(4u+11) = 2x + C_1. \quad \text{--- (2)}$$

$$\text{Now } u = x - 2y.$$

$$\text{(2)} \Rightarrow x - 2y - \frac{1}{4} \ln(4x - 8y + 11) = 2x + C_1.$$

$$\Rightarrow 4x - 8y - \ln(4x - 8y + 11) = 8x + 4C_1,$$

$$\Rightarrow 4x + 8y + \ln(4x - 8y + 11) = C.$$

Ans

EXERCISE 1.3

even Numbered Questions.

(1) Why it is important to add constant of integration immediately when the integration is performed?

Ans Because if one otherwise there is a danger of forgetting to add it or of its getting ignored as a factor during subsequent integrations.

$$(2) \quad yy' + 25x = 0$$

$$\Rightarrow yy' = -25x \Rightarrow \int y dy = \int -25x dx \Rightarrow \frac{y^2}{2} = -\frac{25}{2}x^2 + C$$

$$(3) \quad y' + 3x^2y = 0$$

$$\Rightarrow \frac{y'}{y} = -3x^2 \Rightarrow \int \frac{dy}{y} = \int -3x^2 dx \Rightarrow -\frac{1}{y} = -x^3 - C$$

$$\Rightarrow x^3y + Cy = 0 \quad \text{Ans}$$

$$(6) y' = -ky^2$$

$$\Rightarrow \int \frac{dy}{y^2} = -\int k dx$$

$$Q6 \Rightarrow \frac{-1}{y} = -kx + c$$

$$\Rightarrow kyx + cy = 0 \quad \underline{\text{Ans}}$$

$$(10) y' = (y+4x)^2 \quad (y+4x = u)$$

$$\text{let } y+4x = u$$

$$\Rightarrow y' = u^2$$

$$\begin{aligned} y &= u - 4x \\ \Rightarrow y' &= u' - 4 \end{aligned}$$

$$u' - 4 = u^2$$

$$\Rightarrow \int \frac{du}{u^2 + 4} = \int dx$$

$$\Rightarrow \frac{1}{4} \tan^{-1} \frac{u}{2} = x + c$$

$$\Rightarrow u = 2 \tan(2x + 2c)$$

$$\Rightarrow y = 2 \tan(2x + 2c) - 4x \quad \underline{\text{Ans}}$$

$$(14) y^3 y' + x^3 = 0, y(0) = 1$$

$$\Rightarrow \int y^3 dy = \int -x^3 dx$$

$$\Rightarrow \frac{y^4}{4} = -\frac{x^4}{4} + c$$

$$\Rightarrow x^4 + y^4 = c \quad \underline{\text{Ans}}$$

$$(20) xy' = (y-x)^3 + y$$

$$\Rightarrow xy' = y^3 - 3y^2x + 3yx^2 - x^3 + y$$

$$\Rightarrow y' = y^2(x) - 3y^2 + 3xy - x^2 + \frac{y}{x} \Rightarrow y = ce^{-t} \quad \underline{\text{Ans}}$$

$$\text{let } y/x = u$$

$$\Rightarrow y' = xu' + u$$

$$(8) xy' = 2 + y \quad (y/x = u)$$

$$\Rightarrow y' = 2/x + y/x$$

$$\Rightarrow y' = 1 + y/x \quad (1)$$

$$\text{let } y/x = u \Rightarrow y = xu$$

$$Q8 \Rightarrow y' = xu' + u \quad \text{pulling in (1)}$$

$$\Rightarrow xu' + u = 1 + u$$

$$\Rightarrow xu' = 1$$

$$\Rightarrow \int du = \int \frac{1}{x} dx$$

$$\Rightarrow u = \ln x + \ln c$$

$$\Rightarrow y/x = \ln x$$

$$\Rightarrow y = x \ln x \quad \underline{\text{Ans}}$$

$$(12) y' = -x/y, y(1) = \sqrt{3}$$

$$\Rightarrow y dy = -x dx$$

$$\Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$\Rightarrow x^2 + y^2 = c \quad \underline{\text{Ans}}$$

$$(16) y' = 1 + 4y^2, y(0) = 0$$

$$\Rightarrow \int \frac{dy}{1 + 4y^2} = \int dx$$

$$\Rightarrow \tan^{-1} 2y = x + c$$

$$\Rightarrow y = \frac{1}{2} \tan(x + c) \quad \underline{\text{Ans}}$$

$$(18) dx/dt = -2tx, x(0) = 2.5$$

$$\Rightarrow \int \frac{dx}{x} = \int -2t dt$$

$$\Rightarrow \ln x = -t^2 + \ln c$$

$$\Rightarrow x = ce^{-t^2} \quad \underline{\text{Ans}}$$

$\Rightarrow xu' + u = u^3x - 3u^2x + 3ux^2 - x^2 + u$
 $\Rightarrow u' = u^3x - 3u^2x + 3ux - x$
 $\Rightarrow u' = x(u^3 - 3u^2 + 3u - 1)$
 $\Rightarrow \frac{du}{u^3 - 3u^2 + 3u - 1} = x dx$
 $\Rightarrow \int \frac{du}{(u-1)^3} = \int x dx$
 $\Rightarrow \frac{(u-1)^{-2}}{-2} = \frac{x^2}{2} + C$
 $\Rightarrow \frac{1}{(u-1)^2} = x^2 + C$
 $\Rightarrow -x^2(u-1)^2 - C = 1$
 $\Rightarrow x^2(u-1)^2 + 1 = C$
 $\Rightarrow x^2(\frac{y}{x} - 1)^2 + 1 = C$
 $\Rightarrow x^2(\frac{y^2}{x^2} - 2\frac{y}{x} + 1) + 1 = C$
 $\Rightarrow y^2 - 2xy + 2x^2 = C$

(25) Solve $y' = \frac{1-2y-4x}{1+y+2x}$
 $\Rightarrow y' = \frac{1-2(y+2x)}{1+y+2x}$ — (1)
 Let $u = y+2x \Rightarrow y = u-2x$
 $\Rightarrow y' = u' - 2$, putting in (1)
 $\Rightarrow u' - 2 = \frac{1-2u}{1+u}$
 $\Rightarrow u' = \frac{1-2u}{1+u} + 2$
 $\Rightarrow u' = \frac{1-2u+2+2u}{1+u}$
 $\Rightarrow \int \frac{(1+u)du}{(1+u)^2} = \int 3dx$
 $\Rightarrow \frac{(1+u)^{-1}}{-1} = 3x + C$
 $\Rightarrow (1+u)^{-1} = 3x + C$
 $\Rightarrow \frac{1}{1+u} = 3x + C$
 $\Rightarrow (y+2x)^{-1} + 2y + 4x + 1 = 6x + C$
 $\Rightarrow \frac{1}{y+2x} + 2y + 4x + 1 = 6x + C$

(22) $xy' = y + x \sec(y/x), y(1) = \pi$
 $\Rightarrow y' = \frac{y}{x} + \sec(y/x)$ — (1)
 Let $u = y/x$
 $\Rightarrow y' = u'x + u$, putting in (1)
 $\Rightarrow u'x + u = u + x \sec u$
 $\Rightarrow \int \sec u du = \int x dx$
 $\Rightarrow \int \sec u du = \int dx$
 $\Rightarrow \sin u = x + C$
 $\Rightarrow u = \sin^{-1}(x+C)$
 $\Rightarrow y = x \sin^{-1}(x+C)$ Ans

(24) $y' = f(ax+by+k)$ can be made separable by using a new unknown u
 $u(x) = ax+by+k$. Using this solve $y' = (x+y-2)^{-1}$ — (1)
 Let $u = x+y-2$
 $\Rightarrow y = u - x + 2$
 $\Rightarrow y' = u' - 1$, putting in (1)
 $\Rightarrow u' - 1 = \frac{1}{u}$
 $\Rightarrow \frac{du}{1+u^2} = dx$
 $\Rightarrow \tan^{-1} u = x + C$
 $\Rightarrow u = \tan^{-1}(x+C)$
 $\Rightarrow x+y-2 = \tan^{-1}(x+C)$
 $\Rightarrow y = 2 - x + \tan^{-1}(x+C)$

$$\Rightarrow \int 1 dx + \int u du = \int 3 dx$$

$$\Rightarrow u + \frac{u^2}{2} = 3x + C_1$$

$$\Rightarrow 2u + u^2 = 6x + C$$

$$\Rightarrow 2(y + 2x) + (y + 2x)^2 = 6x + C$$

$$\Rightarrow 2y + 4x + (y + 2x)^2 = 6x + C$$

$$\Rightarrow 2y - 2x + (y + 2x)^2 = C \quad \text{Ans}$$

Exact Differential Equation

By calculus we remember that if a function $u(x, y)$ has continuous partial derivatives then the differential is

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad \text{--- (a)}$$

From this it follows that if $u(x, y) = C$ - constant - then $du = 0$

e.g. if $u = x + 2y^3 = C$ then

$$du = (1 + 6xy^3) dx + 3x^2 y^2 dy = 0$$

$$\Rightarrow y' = \frac{dy}{dx} = - \frac{1 + 6xy^3}{3x^2 y^2} \quad \text{--- (b)}$$

A diff eq that we can solve by going backward. This idea gives a powerful solution method as follows.

A 1st order diff equation of the form

$$M dx + N dy = 0 \quad \text{or} \quad M(x, y) dx + N(x, y) dy = 0 \quad \text{--- (1)}$$

is called an exact diff. equation if the differential $M(x, y) dx + N(x, y) dy$ is exact, i.e., this form is the differential of some

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \quad \text{of some function}$$

$u(x, y)$. Then differential eq (1) can be written as

$$du = 0$$

comparing ① and ② we have

$$\frac{\partial u}{\partial x} = M \quad \& \quad \frac{\partial u}{\partial y} = N$$

$$\Rightarrow \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial M}{\partial y} \quad \& \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial N}{\partial x} \quad \text{③} \quad \text{comparing both}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{--- ④}$$

this condition is not only necessary but also sufficient for ① to be exact differential equation.

If ① is exact, the function $u(x, y)$ can be found by guessing or in the following systematic way. From ④(a) we have by integration.

$$u = \int M dx + k(y). \quad \text{--- ⑤}$$

In this integration, y is to be regarded as a constant, $k(y)$ plays the role of a "constant" of integration. To determine $k(y)$, we derive $\frac{\partial u}{\partial y}$ from ⑤, use ④(b) to get $\frac{dk}{dy}$, and integrate.

$\frac{dk}{dy}$ to get k .

Formula ⑥ was obtained from ④(a). Instead of ④(a) we may equally well use ④(b). Then instead of ⑥ we first have.

$$u = \int N dy + l(x). \quad \text{--- ⑥}^*$$

To determine $l(x)$ we derive $\frac{\partial u}{\partial x}$ from ⑥*, use ④(a) to get $\frac{dl}{dx}$ and integrate to get l .

We illustrate all this by the following typical examples.

EXAMPLE 3 (BOOK)

Solve $(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$

Solution

Step # 01

Test for exactness

Example
I

Our equation is of the form (1) with

$$M = x^3 + 3xy^2, \quad N = 3x^2y + y^3,$$

$$\text{Thus } \frac{\partial M}{\partial y} = 6xy \quad \& \quad \frac{\partial N}{\partial x} = 6xy.$$

From this and fine we see that (7) is exact.

2nd Step: Implicit Solution

From (6) we have

$$u = \int M dx + k(y)$$

$$\Rightarrow u = \int (x^3 + 3xy^2) dx + k(y).$$

$$\Rightarrow u = \frac{x^4}{4} + \frac{3x^2y^2}{2} + k(y). \quad \text{--- (8)}$$

To find $k(y)$, we differentiate this formula w.r.t y and use formula (4b), obtaining

$$\frac{\partial u}{\partial y} = 3x^2y + \frac{dk}{dy}.$$

$$\Rightarrow N = 3x^2y + y^3 = 3x^2y + \frac{dk}{dy}.$$

$$\Rightarrow 3x^2y + y^3 = 3x^2y + \frac{dk}{dy}$$

$$\Rightarrow y^3 = \frac{dk}{dy}.$$

$$\Rightarrow \int y^3 dy = \int dk. \Rightarrow \frac{y^4}{4} = k + c.$$

$$\Rightarrow k = \frac{y^4}{4} + c \quad \text{pulling in (8)}$$

$$\Rightarrow u = \frac{x^4}{4} + \frac{3x^2y^2}{2} + \frac{y^4}{4} + c = 0$$

$$\Rightarrow u = \frac{x^4 + 6x^2y^2 + y^4}{4}$$

$$\Rightarrow x^4 + 6x^2y^2 + y^4 = 4c$$

$$\Rightarrow x^4 + 6x^2y^2 + y^4 = c \quad \text{Ans}$$

Initial Value Problem

Solve $(\sin x \cosh y) dx - (\cos x \sinh y) dy = 0$, $y(0)=3$

Solution

Let $M = \sin x \cosh y$ & $N = -\cos x \sinh y$.

now $\frac{\partial M}{\partial x} = M$ $\frac{\partial M}{\partial y} = \sin x \sinh y$

$\Rightarrow u = \int M dx + k(y)$ $\frac{\partial N}{\partial x} = \sin x \sinh y$

$\Rightarrow u = \int \sin x \cosh y dx + k(y)$ hence it is an exact differential equation.

$\Rightarrow u = -\cos x \cosh y + k(y)$ ①

differentiating w.r.t y

$\Rightarrow \frac{\partial u}{\partial y} = -\cos x \sinh y + \frac{dk}{dy}$

but $\frac{\partial u}{\partial y} = N = -\cos x \sinh y$

$\Rightarrow -\cos x \sinh y = -\cos x \sinh y + \frac{dk}{dy}$

$\Rightarrow \frac{dk}{dy} = 0 \Rightarrow \int dk = \int dy(0)$

$\Rightarrow k = C$

$\Rightarrow k = 0$

Putting in ①

$\Rightarrow u(x, y) = -\cos x \cosh y + 0 = 0$

$\Rightarrow \cos x \cosh y = C$ — ②

now $y(0)=3$ inserting in ②

$\Rightarrow \cos 0 \cosh 3 = C$

$\Rightarrow C = \cosh 3$

$\Rightarrow \cos x \cosh y = \cosh 3$

Integrating Factor

Let we have the equation.

$-y dx + x dy = 0$ — ①

we have $M = -y$ & $N = x$

$\Rightarrow \frac{\partial M}{\partial y} = -1$ and $\frac{\partial N}{\partial x} = 1$

$\Rightarrow \frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}$ hence this equation is

not exact, so we multiply this equation with a factor that is called Integrating factor.

e.g. multiplying (1) by $\frac{1}{y}$.

$$\Rightarrow -\frac{1}{x} dx + \frac{1}{y} dy = 0$$

$$\Rightarrow \int \frac{1}{y} dy = \int \frac{dx}{x} \Rightarrow \ln y = \ln x + \ln c$$

$$\Rightarrow y = cx \Rightarrow c = y/x = u(x, y)$$

or we can multiply it with $\frac{1}{x^2}$ or $\frac{1}{y^2}$ or $\frac{1}{(x^2+y^2)}$

Let this Integrating factor be denoted by F .

In the above example the exact differential equation is

$$FP dx + FQ dy = -\frac{y}{x^2} dx + \frac{x}{x^2} dy = d\left(\frac{y}{x}\right) = 0$$

now for exactness.

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ)$$

$\Rightarrow PF_y + FPy = QFx + FQx$ - In general case this would be complicated and useless. ~~In order~~ In order to make it simple and solvable, we look for an integrating factor depending only on one variable namely x or y . Fortunately in many practical cases there are such factors.

Let $F = F(x)$, thus $F_y = 0$.

$$FP_y = QFx + FQx \quad \text{dividing throughout by } FQ$$

$$\Rightarrow \frac{P_y}{Q} = \frac{Fx}{F} + \frac{Qx}{Q}$$

$$\Rightarrow \frac{F_x}{F} = \frac{P_y}{Q} - \frac{Qx}{Q} = \frac{1}{Q}(P_y - Q_x)$$

$$\frac{1}{F} \frac{dF}{dx} = \frac{1}{Q} (Py - Qx)$$

$$\Rightarrow \int \frac{1}{F} dF = \int \frac{1}{Q} (Py - Qx) dx$$

$$\Rightarrow \ln F = \int \frac{1}{Q} (Py - Qx) dx$$

$$\Rightarrow F(x) = e^{\int \frac{1}{Q} (Py - Qx) dx}$$

$$\Rightarrow F(x) = e^{\int \frac{1}{Q} (\partial P / \partial y - \partial Q / \partial x) dx} \quad \text{--- (2)}$$

(*) Note we have used $\frac{dF}{dx}$ but $\frac{\partial P}{\partial y}$ & $\frac{\partial Q}{\partial x}$ because F depends upon one variable and P and Q depends upon more than one variable.

Integrating Factor (F_x)

In (2) let we denote the right side by R
 $e^{\int R(x) dx}$ such that R depends only on x . Then the integrating factor is

$$F(x) = e^{\int R(x) dx} \quad \text{--- (3)}$$

Integrating Factor (F_y).

Similarly, if $F = F(y)$, then instead of (2) we get

$$\frac{1}{F} \frac{dF}{dy} = \frac{1}{P} (Qx - Py)$$

$$\text{or } \frac{1}{F} \frac{dF}{dy} = \frac{1}{P} (\frac{\partial Q}{\partial x} - \partial P / \partial y) \quad \text{--- (4)}$$

If we denote rightside of (4) by R that depends only on y . Then the Integrating Factor is given by

$$F(y) = e^{\int R(y) dy}$$