DIFFERENTIAL EQUATIONS

EXERCISE 1.3

Problems solved by;

Umair Asghar

NWFP, UET Peshawar

Q # 3 — Q # 23

Q # 3 $y' = 1 + 0.01y^2$

Solution $y' = 1 + 0.01y^2$

$$\Rightarrow \frac{dy}{dx} = 1 + 0.01y^2$$

$$\Rightarrow \frac{1}{(1+0.01y^2)} \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow \int \frac{1}{(1+0.01y^2)} \cdot \frac{dy}{dx} \cdot dx = \int dx$$

($\because$ integrating b/s w.r.t x)

$$\Rightarrow \int \frac{dy}{(1+0.01y^2)} = \int dx$$

$\because \left(\frac{dy}{dx}\right) dx = dy$.

$$\Rightarrow \tan^{-1}(0.1y) = x + c$$

$$\Rightarrow y = 10\tan(x+c).$$ Ans

Q # 5 $y' = xy/2$

Solution

$y' = xy/2$

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{2} \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{x}{2}$$

$$\Rightarrow \int \frac{1}{y} \cdot \frac{dy}{dx} \cdot dx = \int \frac{x}{2} dx$$

($\because$ integrating b/s w.r.t x)

$$\Rightarrow \int \frac{dy}{y} = \frac{1}{2}\int x\,dx$$

$$\Rightarrow \ln y = \frac{1}{2} \cdot \frac{x^2}{2} + c$$

$$\Rightarrow y = e^{x^2/4 + c}$$

$\Rightarrow y = ce^{x^2/4}$ $\because e^c = c$

Q#7 $xy' = y^2 + y$ $(y/x = u)$

Solution

$xy' = y^2 + y.$

$\Rightarrow y' = y(y/x) + (y/x)$ —— ①

Let $y/x = u \Rightarrow y = ux \Rightarrow y' = xu' + u$

putting in ①

$\Rightarrow xu' + \cancel{u} = ux(u) + \cancel{u}$

$\Rightarrow u' = u^2$

$\Rightarrow \frac{du}{dx} = u^2 \Rightarrow \frac{1}{u^2} \cdot \frac{du}{dx} = 1$

$\Rightarrow \int \frac{1}{u^2} \cdot \frac{du}{dx} \cdot dx = \int dx$

$(\because$ integrating b/s w.r.t $x)$

$\Rightarrow \int \frac{du}{u^2} = \int dx.$

$\Rightarrow -\frac{1}{u} = x + c$

$\Rightarrow -\frac{x}{y} = x + c.$ $(\because u = y/x)$

$\Rightarrow -x = y(x+c).$

$\Rightarrow y = -\frac{x}{(x+c)}$

Ans

Solution

Q9 $y' = (x^2+y^2)/xy.$

$\Rightarrow y' = \frac{x}{y} + y/x. \text{ ——— } (1)$

Let $y/x = u \Rightarrow y = ux \Rightarrow y' = xu' + u$

putting in (1)

$\Rightarrow y' = xu' + \cancel{u} = \frac{1}{u} + \cancel{u}.$

$\Rightarrow xu' = \frac{1}{u}.$

$\Rightarrow uu' = \frac{1}{x}.$

$\Rightarrow u\frac{du}{dx} = \frac{1}{x}.$

$\Rightarrow \int u\frac{du}{dx}.dx = \int \frac{1}{x} dx$

(∵ integrating b/s w.r.t x)

$\Rightarrow \int u\, du = \int \frac{1}{x} dx$

$(\because (\frac{du}{dx})\, dx = du)$

$\Rightarrow \frac{u^2}{2} = \ln x + \cancel{\ln} c_1$

$\Rightarrow u^2 = 2\ln x + \cancel{\ln} c \quad (\because c = 2c_1)$

$\Rightarrow \frac{y^2}{x^2} = 2\ln x + c \quad (\because u = y/x).$

$\Rightarrow y^2 = 2x^2 \ln x + cx^2.$

$\Rightarrow y = x\sqrt{2\ln x + c}$ **Ans**

Solution.

Q11

$$y' + \operatorname{cosec} y = 0.$$

$$\Rightarrow dy/dx = -\operatorname{cosec} y.$$

$$\Rightarrow \frac{1}{\operatorname{cosec} y} \cdot \frac{dy}{dx} = -1$$

$$\Rightarrow \int \frac{1}{\operatorname{cosec} y} \cdot \frac{dy}{dx} \cdot dx = \int -1\, dx.$$

$(\because$ integrating b/s w.r.t $x)$

$$\Rightarrow \int \sin y\, dy = -\int dx$$

$$\left(\because \left(\frac{dy}{dx}\right) dx = dy\right)$$

$$\Rightarrow -\cos y = -x + c_1$$

$$\Rightarrow y = \cos^{-1}(x + c). \quad \because -c_1 = c$$

Ans

Q # 13 $xy' + y = 0$, $y(2) = -2$

Solution

$$\Rightarrow x\frac{dy}{dx} = -y$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = -\frac{1}{x}.$$

$$\Rightarrow \int \frac{1}{y} \frac{dy}{dx} \cdot dx = \int -\frac{1}{x} dx$$

$(\because$ integrating b/s w.r.t $x)$

$$\Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\Rightarrow \ln y = -\ln x + \ln c.$$

$$\Rightarrow y = c/x \quad \text{——} (1)$$

Now $y(2) = -2$, Given.

putting in (1)

Substituting in (1)

$$\Rightarrow \quad y = -4/x \Rightarrow xy = -4$$

Ans

Q # 15

$$e^{x} y' = 2(x+1)y^{2} \quad , \quad y(0) = 1/6$$

Solution

$$e^{x}\frac{dy}{dx} = 2(x+1)y^{2}.$$

$$\Rightarrow \frac{1}{y^{2}}\frac{dy}{dx} = \frac{2(x+1)}{e^{x}}$$

(Integrating b/s w.r.t x)

$$\Rightarrow \int \frac{1}{y^{2}}\frac{dy}{dx}\cdot dx = \int \frac{2(x+1)}{e^{x}}\, dx.$$

$$\Rightarrow \int \frac{dy}{y^{2}} = 2\int e^{-x}(x+1)\, dx.$$

$$\Rightarrow -\frac{1}{y} = +2\left(-e^{-x}(x+1) - \int -e^{-x}dx\right) + c$$

$$\Rightarrow -\frac{1}{y} = -2\left(e^{-x}(x+1) + (-e^{-x})\right) + c$$

$$\Rightarrow \frac{1}{y} = 2\left(e^{-x}(x+1) + e^{-x}\right) + c. \quad \text{—— (1)}$$

Now $y(0) = 1/6$

putting in (1)

$$\Rightarrow 6 = 2\left(e^{0}(0+1) + e^{0}\right) + c.$$

$$\Rightarrow 6 = 4 + c$$

$$\Rightarrow c = 2$$

Substituting in (1)

$$\Rightarrow y = \frac{1}{e^{-x}((2x+2)+2)+2}$$

$$2 + (2x+4)e^{-x}$$

Ans

Q # 17

$$y' \cosh^2 x - \sin^2 y = 0 \quad , \quad y(0) = \pi/2$$

Solution

$$\Rightarrow \frac{dy}{dx} \cosh^2 x = \sin^2 y.$$

$$\Rightarrow \frac{1}{\sin^2 y} \cdot \frac{dy}{dx} = \frac{1}{\cosh^2 x}$$

$$\Rightarrow \operatorname{cosec}^2 y \cdot \frac{dy}{dx} = \operatorname{sech}^2 x.$$

integrating b/s w.r.t x.

$$\Rightarrow \int \operatorname{cosec}^2 y \cdot \frac{dy}{dx} \cdot dx = \int \operatorname{sech}^2 x \, dx.$$

$$\Rightarrow \int \operatorname{cosec}^2 y \, dy = \int \operatorname{sech}^2 x \, dx.$$

$$\because \left(\frac{dy}{dx}\right) dx = dy.$$

$$\Rightarrow -\cot y = \tanh x + c \quad — (1)$$

Now $y(0) = \pi/2$ putting in (1)

$$\Rightarrow -\cot \pi/2 = \tanh 0 + c.$$

$$\Rightarrow 0 = c$$

$$\Rightarrow c = 0$$ substituting in (1)

$$\Rightarrow -\cot y = \tanh x + 0$$

$$\Rightarrow y = \cot^{-1}(-\tanh x).$$

Ans

(L & R are constants)

Solution

Q19

$\Rightarrow L\,(dI/dt) = -RI$

$\Rightarrow \frac{1}{I}\cdot dI/dt = -\frac{R}{L}$

Integrating b/s w.r.t, "t"

$\Rightarrow \int \frac{1}{I}\cdot\frac{dI}{dt}\cdot dt = -\frac{R}{L}\int dt$

$\Rightarrow \int \frac{1}{I}\,dI = -\frac{R}{L}\int dt$

$\because \left(\frac{dI}{dt}\right)dt = dI$

$\Rightarrow \ln I = -\frac{R}{L}t + \ln C$

$\Rightarrow I/C = e^{-(R/L)t}$

$\Rightarrow I = Ce^{-(R/L)t}$ —— (1)

Now $I(0) = I_0$ putting in (1)

$\Rightarrow I_0 = C$ substituting in (1)

$\Rightarrow I = I_0 e^{-(R/L)t}$

Ans

Solution

Q21 $\Rightarrow y' = \frac{y}{x} + 3x^3\cos^2(y/x)$ — (1)

Let $y/x = u \Rightarrow y = ux$.

$\Rightarrow y' = xu' + u$. putting in (1)

$\Rightarrow xu' + u = u + 3x^3\cos^2 u$.

$\Rightarrow u' = 3x^2\cos^2 u$.

$\Rightarrow \frac{du}{dx} = 3x^2\cos^2 u$.

$\Rightarrow \frac{1}{\cos^2 u} \cdot \frac{du}{dx} = 3x^2$.

$\Rightarrow \sec^2 u \cdot \frac{du}{dx} = 3x^2$.

integrating b/s w.r.t x.

$\Rightarrow \int \sec^2 u \cdot \frac{du}{dx} \cdot dx = \int 3x^2 dx$.

$\Rightarrow \int \sec^2 u \cdot du = 3\int x^2 dx$.

$\because \left(\frac{du}{dx}\right)(dx) = du$.

$\Rightarrow \tan u = 3\frac{x^3}{3} + c$

$\Rightarrow \tan u = x^3 + c$.

$\Rightarrow \tan y/x = x^3 + c$

$\because u = y/x$.

$\Rightarrow y = x\tan^{-1}(x^3 + c)$ — (2)

$\Rightarrow 0 = \tan^{-1}(1+c).$

$\Rightarrow c+1 = \tan 0$

$\Rightarrow c = -1$

Substituting in (2)

$\Rightarrow y = x\tan^{-1}(x^3-1).$

Ans

Q#23 $xyy' = 2y^2 + 4x^2, \quad y(2) = 4$

Solution

$xyy' = 2y^2 + 4x^2.$

$\Rightarrow y' = 2\,y/x + 4\,x/y$ —— (1)

Let $u = y/x \Rightarrow y = ux \Rightarrow y' = xu' + u$

putting in (1)

$\Rightarrow xu' + u = 2u + \frac{4}{u}.$

$\Rightarrow xu' = u + \frac{4}{u}.$

$\Rightarrow xu' = \frac{u^2+4}{u}.$

$\Rightarrow \frac{u}{u^2+4}\frac{du}{dx} = \frac{1}{x}$

integrating b/s w.r.t x.

$\Rightarrow \int \frac{u}{u^2+4}\cdot\frac{du}{dx}\cdot dx = \int \frac{1}{x}\,dx.$

$\Rightarrow \frac{1}{2}\cdot\int \frac{2u}{u^2+4}\,du = \int \frac{1}{x}\,dx.$

$\because \left(\frac{du}{dx}\right)dx = du$

$\Rightarrow \ln(u^2+4)^{1/2} = \ln(cx)$

$\Rightarrow (u^2+4)^{1/2} = cx$

$\Rightarrow \left(\frac{y^2}{x^2}+4\right)^{1/2} = cx \quad (\because u = y/x)$

$\Rightarrow \frac{y^2}{x^2}+4 = c^2x^2$

$\Rightarrow y^2+4x^2 = c^2x^4$ ——— (2)

Now $y(2) = 4$.

$\Rightarrow 16 + 4(4) = c^2(16)$

$\Rightarrow 32 = c^2(16)$

$\Rightarrow c^2 = 2 \Rightarrow c = \sqrt{2}$ putting in (2)

$\Rightarrow y^2 + 4x^2 = 2x^4$.

$\Rightarrow y^2 = 2x^4 - 4x^2$.

$\Rightarrow y = \sqrt{2x^4 - 4x^2}$

Ans

Many 1st order differential equations reduce to the form:

(1) $g(y)\,y' = f(x)$ — ① by algebric manipulations.

$\because y' = dy/dx$

$\therefore g(y)\frac{dy}{dx} = f(x) \Rightarrow g(y)dy = f(x)dx$ — ②

Such an equation is called seperable equation, because in ② the variables x and y are seperated so that x appears only on the right and y on the left.

To solve (1), integrate both sides w.r.t x obtaining

$\int g(y)\frac{dy}{dx}dx = \int f(x)dx + c.$

Now on the left we can switch to y as the variable of integration.

By calculus $(dy/dx)(dx) = dy$. so that we get

$\int g(y)\,dy = \int f(x)\,dx + c$ — ③

If we assume that f and g are continuous functions, the integrals in ③ will exist, and by evaluating these integrals we obtain the general solution of ①.

REDUCTION TO SEPERABLE FORM

Certain differential equations are not seperable but can be made seperable by the introduction of new unknown function, as in the following example.

EXAMPLE 6 (BOOK)

Solve $2xyy' = y^2 - x^2$

$\Rightarrow y' = \frac{y^2}{2xy} - \frac{x^2}{2xy} = \frac{1}{2}\left(\frac{y}{x} - x/y\right)$ — ①

putting $y/x = u \Rightarrow y = ux \Rightarrow y' = xu' + u$.

putting in (1)

$\Rightarrow y' = xu' + u = \frac{1}{2}(u - \frac{1}{u})$.

$\Rightarrow u'x = \frac{1}{2}u - \frac{1}{2u} - u = -\frac{1}{2u} - \frac{1}{2}u$.

$\Rightarrow u'x = -\frac{1}{2}(\frac{1}{u} + u) = -\frac{1}{2}(\frac{1+u^2}{u})$.

$\Rightarrow u'x = -\frac{1}{2}(\frac{1+u^2}{u})$.

$\Rightarrow \int \frac{2u}{1+u^2}\,du = \int -\frac{dx}{x}$

$\Rightarrow \ln(1+u^2) = -\ln x + \ln c$.

$\Rightarrow 1+u^2 = c/x$

Now putting $u = y/x$.

$\Rightarrow 1 + y^2/x^2 = c/x. \Rightarrow x^2 + y^2 = cx$

$\Rightarrow x^2 - cx + \frac{c^2}{4} - \frac{c^2}{4} + y^2 = 0$

$\Rightarrow (x - c/2)^2 + y^2 = c^2/4$.

Ans

Transformations.

Sometimes a transformation $v = ay + bx + k$ may lead to a seperable differential equation. as in the following example.

Example 7 (book)

$(2x - 4y + 5)y' + x - 2y + 3 = 0$. —(1)

put $x - 2y = u \Rightarrow y = \frac{x - u}{2} \Rightarrow y' = \frac{1 - u'}{2}$.

putting in (1)

$\Rightarrow (2u + 5)(\frac{1 - u'}{2}) + u + 3 = 0$

$\Rightarrow (2u+5)(1-u') = -2u - 6$.

$\Rightarrow (4u+10)u' = 8u+22.$

$\Rightarrow \frac{4u+10}{4u+11}du = 2dx \Rightarrow \frac{4u+11-1}{4u+11}du = 2dx$

$\Rightarrow \int du - \int \frac{1}{4u+11}du = \int 2dx.$

$\Rightarrow u - \frac{1}{4}\ln(4u+11) = 2x + C_1 . \quad -(2)$

Now $u = x-2y.$

$(2) \Rightarrow x-2y-\frac{1}{4}\ln(4x-8y+11) = 2x+C_1.$

$\Rightarrow 4x-8y-\ln(4x-8y+11) = 8x+4C_1$

$\Rightarrow 4x+8y+\ln(4x-8y+11) = C.$

Ans

EXERCISE 1.3

even Numbered Questions.

(1) Why it is important to add constant of integration immediately when the integration is performed?

Ans Because if one

otherwise there is a danger of forgetting to add it or of its getting ignored as a factor during subsequent integrations.

(2) $yy'+25x=0$

$\Rightarrow yy' = 25x \Rightarrow \int y\,dy = \int 25x\,dx \Rightarrow \frac{y^2}{2} = \frac{25}{2}x^2 + c$

(4) $y'+3x^2y^2=0$

$\Rightarrow y' = -3x^2y^2 \Rightarrow \int \frac{dy}{y^2} = \int -3x^2\,dx \Rightarrow -\frac{1}{y} = -x^3 - c$

$\Rightarrow x^3y + cy = 0$ Ans

(6) $y' = -ky^2$

$\Rightarrow \int \frac{dy}{y^2} = -\int k\,dx$

Q6 $\Rightarrow \frac{-1}{y} = -kx + c$

$\Rightarrow kyx + cy = 0$ Ans

(10) $y' = (y+4x)^2 \quad (y+4x = u)$

Let $y + 4x = u$.

$\Rightarrow y' = u^2$.

$y = u - 4x$

$\Rightarrow y' = u' - 4$.

$u' - 4 = u^2$

$\Rightarrow \int \frac{du}{u^2+4} = \int dx$

$\Rightarrow \frac{1}{2}\tan^{-1}\frac{u}{2} = x + c$

$\Rightarrow u = 2\tan(2x + 2c)$

$\Rightarrow y = 2\tan(2x+2c) - 4x$ Ans

(14) $y^3 y' + x^3 = 0, \quad y(0) = 1$

$\Rightarrow \int y^3 dy = \int -x^3 dx$

$\Rightarrow \frac{y^4}{4} = -\frac{x^4}{4} + c$

$\Rightarrow x^4 + y^4 = c$. Ans

(20) $xy' = (y-x)^3 + y$

$\Rightarrow xy' = y^3 - 3y^2x + 3yx^2 - x^3 + y$

$\Rightarrow y' = y^2(y/x) - 3y^2 + 3xy - x^2 + y/x$ — (1)

Put $y/x = u$

$\Rightarrow y' = xu' + u$. putting in (1)

(8) $xy' = x + y \quad (y/x = u)$

$\Rightarrow y' = x/x + y/x$.

$\Rightarrow y' = 1 + y/x$ — (1)

Q8 Let $y/x = u \Rightarrow y = xu$

$\Rightarrow y' = xu' + u$. putting in (1)

$\Rightarrow xu' + u = 1 + u$.

$\Rightarrow xu' = 1$

$\Rightarrow \int du = \int \frac{1}{x} dx$

$\Rightarrow u = \ln x + \ln c$.

$\Rightarrow y/x = \ln cx$.

$\Rightarrow y = x \ln cx$. Ans

(12) $y' = -x/y, \quad y(1) = \sqrt{3}$

$\Rightarrow \int y\,dy = \int -x\,dx$

$\Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + c$

$\Rightarrow x^2 + y^2 = c$ Ans

(16) $y' = 1 + 4y^2, \quad y(0) = 0$

$\Rightarrow \int \frac{dy}{1+4y^2} = \int dx$

$\Rightarrow \tan^{-1} 2y = x + c$

$\Rightarrow y = \frac{1}{2}\tan(x + c)$ Ans

(18) $dr/dt = -2tr, \quad r(0) = 2.5$

$\Rightarrow \int \frac{dr}{r} = \int -2t\,dt$

$\Rightarrow \ln r = -t^2 + \ln c$.

$\Rightarrow r = ce^{-t^2}$ Ans

$\Rightarrow xu' + u = u^3x^2 - 3u^2x^2 + 3ux^2 - x^2 + u$

$\Rightarrow u' = u^3x - 3u^2x + 3ux - x$

$\Rightarrow u' = x(u^3 - 3u^2 + 3u - 1)$

$\Rightarrow \dfrac{du}{u^3 - 3u^2 + 3u - 1} = x\,dx.$

$\Rightarrow \int \dfrac{du}{(u-1)^3} = \int x\,dx$

$\Rightarrow \dfrac{(u-1)^{-2}}{-2} = \dfrac{x^2}{2} + c$

$\Rightarrow \dfrac{-1}{(u-1)^2} = x^2 + c$

$\Rightarrow -x^2(u-1)^2 - c = 1$

$\Rightarrow x^2(u-1)^2 + 1 = c$

$\Rightarrow x^2(y/x - 1)^2 + 1 = c$

$\Rightarrow x^2(y^2/x^2 - 2y/x + 1) + 1 = c.$

$\Rightarrow y^2 - 2xy + 2x^2 = c$

(25) solve $y' = \dfrac{1 - 2y - 4x}{1 + y + 2x}$

$\Rightarrow y' = \dfrac{1 - 2(y + 2x)}{1 + y + 2x}$ — (1)

Let $u = y + 2x \Rightarrow y = u - 2x$

$\Rightarrow y' = u' - 2.$ putting in (1)

$\Rightarrow u' - 2 = \dfrac{1 - 2u}{1 + u}$

$\Rightarrow u' = \dfrac{1 - 2u}{1 + u} + 2.$

$\Rightarrow u' = \dfrac{1 - 2u + 2 + 2u}{1 + u}$

~~$\Rightarrow \int (1+u)\,du = \int 3\,dx$~~

~~$\Rightarrow \dfrac{(1+u)^2}{2} = 3x + c,$~~

~~$\Rightarrow (1+u)^2 = 6x + c$~~

$\Rightarrow u^2 + 2u + 1 = 6x + c$

$\Rightarrow (y+2x)^2 + 2y + 4x + 1 = 6x + c$

$\Rightarrow y^2 + 4xy + 4x^2 + 2y + 1 = 2x + c.$

(22) $xy' = y + x\sec(y/x),\ y(1) = \pi$

$\Rightarrow y' = y/x + x\sec(y/x)$ — (1)

Q22 Let $u = y/x.$

$\Rightarrow y' = u'x + u.$ putting in (1)

$\Rightarrow u'x + u = u + x\sec u$

$\Rightarrow \int \dfrac{du}{\sec u} = \int 1\,dx$

$\Rightarrow \int \cos u\,du = \int dx.$

$\Rightarrow \sin u = x + c.$

$\Rightarrow u = \sin^{-1}(x + c)$

$\Rightarrow y = x\sin^{-1}(x + c)$ Ans

(24) $y' = f(ax + by + k)$ **can be made seperable by using a new unknown ft** $u(x) = ax + by + k$. **Using this solve** $y' = (x + y - 2)^2$ — (1)

Let $u = x + y - 2.$

$\Rightarrow y = u - x + 2.$

$\Rightarrow y' = u' - 1$ putting in (1)

$\Rightarrow u' - 1 = u^2.$

$\Rightarrow \dfrac{du}{1 + u^2} = dx.$

$\Rightarrow \tan^{-1}u = x + c.$

$\Rightarrow u = \tan^{-1}(x + c)$

$\Rightarrow x + y - 2 = \tan^{-1}(x + c)$

$\Rightarrow y = 2 - x + \tan^{-1}(x + c)$

$\Rightarrow \int 1\,du + \int u\,du = \int 3\,dx$.

$\Rightarrow u + \frac{u^2}{2} = 3x + c_1$

$\Rightarrow 2u + u^2 = 6x + c$.

$\Rightarrow 2(y+2x) + (y+2x)^2 = 6x + c$.

$\Rightarrow 2y + 4x + (y+2x)^2 = 6x + c$

$\Rightarrow 2y - 2x + (y+2x)^2 = c$ Ans

Exact Differential EQUATION

By calculus we remember that if a function $u(x,y)$ has continuous partial derivatives. Then its differential is

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ —— (a)

From this it follows that if $u(x,y) = c$ = constant then $du = 0$.

e.g. if $u = x + x^2y^3 = c$ then.

$du = (1 + 2xy^3)dx + 3x^2y^2dy = 0$.

$\Rightarrow y' = \frac{dy}{dx} = -\frac{1+2xy^3}{3x^2y^2}$ —— (b)

A diff eq that we can solve by going backward. This idea gives a powerful solution method as follows.

A 1st order diff equation of the form.

$Mdx + Ndy = 0$ or $M(x,y)dx + N(x,y)dy = 0$ — (1)

is called an exact diff. equation if the differential $M(x,y)dx + N(x,y)dy$ is exact, i.e, this form is the differential ~~of some~~.

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ — (2) of some function $u(x,y)$. Then differential eq (1) can be written as

$du = 0$.

comparing (1) and (2) we have.

$\frac{\partial u}{\partial x} = M$ & $\frac{\partial u}{\partial y} = N$.

$\Rightarrow \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial M}{\partial y}$ (a) & $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial N}{\partial x}$ (b) — (4) comparing both

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ —— (5)

This condition is not only necessary but also sufficient for (1) to be exact differential equation.

If (1) is exact, the function $u(x,y)$ can be found by guessing or in the following systematic way. From 4(a) we have by integration.

$u = \int M dx + k(y)$. —— (6)

In this integration, y is to be regarded as a constant, $k(y)$ plays the role of a "constant" of integration. To determine $k(y)$, we derive $\frac{\partial u}{\partial y}$ from (6), use (4b) to get $\frac{dk}{dy}$, and integrate $\frac{dk}{dy}$ to get k.

Formula (6) was obtained from (4a). Instead of (4a) we may equally well use 4(b). Then instead of (6) we first have.

$u = \int N dy + l(x)$. —— (6)*

To determine $l(x)$ we derive $\frac{\partial u}{\partial x}$ from (6*), use (4a) to get dl/dx and integrate to get l.

We illustrates all this by the following typical examples.

EXAMPLE 1 (BOOK)

Solve $(x^3+3xy^2)dx+(3x^2y+y^3)dy=0$

Solution

Step # 01 Test for exactness

Example I

Our equation is of the form (1) with

$M=x^3+3xy^2$, $N=3x^2y+y^3$,

Thus $\frac{\partial M}{\partial y}=6xy$ & $\frac{\partial N}{\partial x}=6xy$.

From this and five we see that (7) is exact.

2nd Step Implicit Solution

From (6) we have:

$u=\int M\,dx+k(y)$

$\Rightarrow u=\int(x^3+3xy^2)dx+k(y)$.

$\Rightarrow u=\frac{x^4}{4}+\frac{3x^2y^2}{2}+k(y)$ —— (8)

To find $k(y)$, we differentiate this formula w.r.t y and use formula (4b), obtaining

$\frac{\partial u}{\partial y}=3x^2y+\frac{dk}{dy}$.

$\Rightarrow N=3x^2y+y^3=3x^2y+\frac{dk}{dy}$.

$\Rightarrow \cancel{3x^2y}+y^3=\cancel{3x^2y}+\frac{dk}{dy}$

$\Rightarrow y^3=\frac{dk}{dy}$.

$\Rightarrow \int y^3dy=\int dk \Rightarrow \frac{y^4}{4}=k+c$.

$\Rightarrow k=\frac{y^4}{4}+c$ putting in (8)

$\Rightarrow u=\frac{x^4}{4}+\frac{3x^2}{2}y^2+\frac{y^4}{4}+c=0$

$\Rightarrow c=\frac{x^4+6x^2y^2+y^4}{4}$

$\Rightarrow x^4+6x^2y^2+y^4=4c$

$\Rightarrow x^4+6x^2y^2+y^4=c$ Ans

Initial Value Problem:

Solve $(\sin x \cosh y)dx - (\cos x \sinh y)dy = 0$, $y(0) = 3$

solution

Let $M = \sin x \cosh y$ & $N = -\cos x \sinh y$.

$\frac{\partial M}{\partial y} = \sin x \sinh y$

& $\frac{\partial N}{\partial x} = \sin x \sinh y$.

hence it is an exact differential equation.

Now $\frac{\partial u}{\partial x} = M$.

e.g. 1 $\Rightarrow u = \int M dx + k(y)$

$\Rightarrow u = \int \sin x \cosh y \, dx + k(y)$

$\Rightarrow u = -\cos x \cosh y + k(y)$ — (1)

differentiating w.r.t y

$\Rightarrow \frac{\partial u}{\partial y} = -\cos x \sinh y + \frac{dk}{dy}$

but $\frac{\partial u}{\partial y} = N = -\cos x \sinh y$.

$\Rightarrow -\cos x \sinh y = -\cos x \sinh y + \frac{dk}{dy}$

$\Rightarrow \frac{dk}{dy} = 0 \Rightarrow \int dk = \int dy(0)$

$\Rightarrow k = c$.

$\Rightarrow k = c$.

putting in (1)

$\Rightarrow u(x,y) = -\cos x \cosh y + c = 0$.

$\Rightarrow \cos x \cosh y = c$ — (2)

Now $y(0) = 3$. inserting in (2)

$\therefore \Rightarrow \cos 0 \cosh 3 = c$

$\Rightarrow c = \cosh 3$

$\therefore \cos x \cosh y = \cosh 3$.

Ans

INTEGRATING FACTOR

Let we have the equation.

$-y dx + x dy = 0$ — (1)

we have $M = -y$ & $N = x$.

$\Rightarrow \frac{\partial M}{\partial y} = -1$ and $\frac{\partial N}{\partial x} = 1$

$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ hence this equation is not exact, to make it exact we multiply this equation with a factor that is called Integrating factor.

e.g multiplying ① by $\frac{1}{xy}$.

$$\Rightarrow -\frac{1}{x}dx + \frac{1}{y}dy = 0$$

$$\Rightarrow \int \frac{1}{y}dy = \int \frac{dx}{x} \Rightarrow \ln y = \ln x + \ln c$$

$$\Rightarrow y = cx \Rightarrow c = y/x = u(x,y).$$

or we can multiply it with $\frac{1}{x^2}$ or $\frac{1}{y^2}$ or $\frac{1}{(x^2+y^2)}$

Let this Integrating factor be denoted by F.

∴ In the above example the exact differential equation is

$$FP\,dx + FQ\,dy = \frac{-y\,dx + x\,dy}{x^2} = d(y/x) = 0$$

Now for exactness.

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ).$$

$\Rightarrow PF_y + FP_y = QF_x + FQ_x$. In General case this would be complicated and useless so In order to make it simple and solveable, we look for an integrating factor depending only on one variable namely x or y, fortunately in many practical cases there are such factors.

So let $F = F(x)$, Thus $F_y = 0$.

∴ $FP_y = QF_x + FQ_x$. ÷ing throughout by FQ

$$\Rightarrow \frac{P_y}{Q} = \frac{F_x}{F} + \frac{Q_x}{Q}.$$

$$\Rightarrow \frac{F_x}{F} = \frac{P_y}{Q} - \frac{Q_x}{Q} = \frac{1}{Q}(P_y - Q_x)$$

$\Rightarrow \frac{1}{F}\frac{dF}{dx} = \frac{1}{Q}(P_y - Q_x).$

$\Rightarrow \int \frac{1}{F} dF = \int \frac{1}{Q}(P_y - Q_x)\,dx$

$\Rightarrow \ln F = \int \frac{1}{Q}(P_y - Q_x)$

$\Rightarrow F_{(x)} = e^{\int \frac{1}{Q}(P_y - Q_x)\,dx}.$

$\Rightarrow F_{(x)} = e^{\int \frac{1}{Q}(\partial P/\partial y - \partial Q/\partial x)\,dx}$ ——— (2)

(*) Note we have used $\frac{dF}{dx}$ but $\frac{\partial P}{\partial y}$ & $\frac{\partial Q}{\partial x}$ because F depends upon one variable and P and Q depends upon more than one variable.

Integrating Factor (Fx)

In (2) let we denote the right side by R $e^{\int R(x)\,dx}$. such that R depends only on x, then the integrating factor is.

$F_{(x)} = e^{\int R(x)\,dx}.$ ——— (3)

Integrating Factor (Fy).

Similarly, if $F = F(y)$, then instead of (2) we get.

$\frac{1}{F}\frac{dF}{dy} = \frac{1}{P}(Q_x - P_y)$

or $\frac{1}{F}\frac{dF}{dy} = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \partial P/\partial y\right)$. — (4)

If we denote rightside of (4) by R. that depends only on y. then the Integrating Factor is given by.

$F(y) = e^{\int R(y)\,dy}.$