DIFFERENTIAL EQUATIONS

EXERCISE 2.1

Problems solved by;

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REDUCTION OF LINEAR OR NONLINEAR EQUATIONS TO FIRST ORDER

A general second order differential equation of the form $F(x, y', y'') = 0$, involving y'', given function of x, and perhaps y and y' can be reduced to first order. This is possible (without knowledge of the solution) if y does not occur explicitly or x does not occurs explicitly.

Solve

(3) $y'' = y'$ —(1) $\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(z)$

Let $y' = z \Rightarrow y'' = z' \Rightarrow y'' = \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} = z\frac{dz}{dy}$

$\therefore y'' = z\frac{dz}{dy}$ putting in (1)

$\Rightarrow z\frac{dz}{dy} = z \Rightarrow z' = z \Rightarrow \frac{dz}{dx} = z$

$\Rightarrow \int z^{-1} dz = \int dx \Rightarrow \ln|z| = x + c_1 \Rightarrow z = c_1 e^{x}$.

$\Rightarrow y' = c_1 e^{x} \Rightarrow \int dy = \int c_1 e^{x} dx \Rightarrow y = c_1 e^{x} + c_2$

OR It can also be solved as follows to get the other sol

$y'' - y' = 0 \Rightarrow u' - u = 0$

Here $p = -1 \Rightarrow U = \frac{e^{-\int p dx}}{y_1^2} = \frac{e^{\int dx}}{y_1^2} = \frac{e^{x}}{y_1^2}$

Now $y_1 = e^{x}$ is a solution of (1)

$\therefore U = \frac{e^{x}}{e^{2x}} = e^{-x}$.

$\therefore u = \int U dx = \int e^{-x} dx = -e^{-x}$.

$\therefore y_2 = u y_1 = -e^{-x} e^{x} = -1$.

(4) $2xy'' = 3y'$ —(1)

Let $y' = z \Rightarrow y'' = z' = \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$

$\Rightarrow y'' = z\frac{dz}{dy}$ ~~putting in (2)~~ putting in (1)

$\Rightarrow \int \frac{dz}{z} = \frac{3}{2}\int \frac{dx}{x} \Rightarrow \ln z = \frac{3}{2}\ln x + \ln c \Rightarrow z = c x^{3/2}$

$\Rightarrow y' = c x^{3/2} \Rightarrow y = 2c\frac{x^{5/2}}{5} + d$ Ans

(5) $yy'' = 2y'^2 \Rightarrow y'' = \frac{2y'^2}{y}$ —— ①

Let $y' = z \Rightarrow y'' = z' = \frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx}$

$\Rightarrow y'' = z\frac{dz}{dy}$ putting in ①

$\Rightarrow z\frac{dz}{dy} = \frac{2z^2}{y} \Rightarrow \int \frac{dz}{z} = 2\int \frac{dy}{y}$

$\Rightarrow \ln z = \ln y^2 + \ln c_1 \Rightarrow z = c_1 y^2$

$\Rightarrow y' = c_1 y^2 \Rightarrow \int y^{-2}dy = c_1\int dx$

$\Rightarrow \frac{y^{-2+1}}{-2+1} = c_1 x + c_2$

$\Rightarrow \frac{y^{-1}}{-1} = c_1 x + c_2 \Rightarrow \frac{1}{y} = -c_1 x + c_2$

$\Rightarrow y = \frac{-1}{(c_1 x + c_2)}$ Ans

(6) $xy'' + 2y' + xy = 0 \quad , y_1 = (\sin x)/x$

$\Rightarrow y'' + \frac{2y'}{x} + y = 0$ —— ①

Here $P = 2/x \Rightarrow \int P dx = 2\ln x$ & $e^{-\int P dx} = e^{\ln x^{-2}}$

$\therefore \; U = \frac{e^{-\int P dx}}{y_1^2} = \frac{x^{-2}}{y_1^2} = \frac{1/x^2}{\sin^2 x/x^2} = \operatorname{cosec}^2 x$

So

$u = \int U dx = \int \operatorname{cosec}^2 x\, dx = -\cot x$

As $y_2 = uy_1 \Rightarrow y_2 = -\cot x \frac{\sin x}{x} = -\frac{\cos x}{x}$

Ans

Sol Let $y' = z \Rightarrow y'' = z' = \frac{dz}{dx} = z\frac{dz}{dy}$, putting in ①

$\Rightarrow z\frac{dz}{dy} + e^y z^3 = 0 \Rightarrow z\frac{dz}{dy} = -e^y z^3$

$\Rightarrow \int \frac{dz}{z^2} = -\int e^y dy \Rightarrow -z^{-1} = -e^y + c.$

$\Rightarrow \frac{1}{z} = e^y + c$

$\Rightarrow z = \frac{1}{e^y + c} \Rightarrow y' = \frac{1}{e^y + c}$

$\Rightarrow \int (c + e^y) dy = dx$

$\Rightarrow cy + e^y = x + d. \Rightarrow x = e^y + cy + d.$

Ans

⑧ $xy'' + y' = 0$ —— ①

$\Rightarrow y'' + \frac{y'}{x} = 0.$

To find y.

Let $y' = z$ in ① $\Rightarrow y'' = z' = \frac{dz}{dx}.$

putting in ①

$\Rightarrow x\frac{dz}{dx} = -z$

$\Rightarrow \int \frac{1}{z} dz = -\int \frac{dx}{x}. \Rightarrow \ln z = -\ln x + \ln c.$

$\Rightarrow z = c/x \Rightarrow y' = c/x \Rightarrow \int dy = c\int \frac{dx}{x}$

$\Rightarrow y = c\ln x + d.$

Ans

⑨ $x^2y'' - 5xy' + 9y = 0$, $y_1 = x^3$

$\Rightarrow y'' - \frac{5y'}{x} + \frac{9y}{x^2} = 0$ —— ①

$\Rightarrow \int 5/x \, dx = 5\ln x.$

$\& \; e^{-\int P dx} = e^{\ln x^5} = x^5.$ putting in (2)

$\Rightarrow U = \frac{x^5}{x^6} \Rightarrow U = x^{-1}$

$\therefore u = \int U dx = \int x^{-1} dx = \ln x + c$

$\Rightarrow y_2 = u y_1 = \ln x (x^3) = x^3 \ln x$

Ans

(10) $y'' + (1+y^{-1}) y'^2 = 0$ — (1)

Let $y' = z \Rightarrow y'' = z' \Rightarrow y'' = z \frac{dz}{dy}$

putting in (1).

$\Rightarrow z \frac{dz}{dy} + (1+y^{-1}) z^2 = 0$

$\Rightarrow \int \frac{dz}{z} = -\int (1+y^{-1}) dy \Rightarrow \ln z = -(y + \ln y) + c$

$\Rightarrow z = e^{-y - \ln y + c}$

$\Rightarrow z = c e^{-y} \cdot e^{-\ln y} = c e^{-y} y^{-1} = \frac{c e^{-y}}{y}$

$\Rightarrow z = \frac{c}{y e^{y}} \Rightarrow y' = \frac{c}{y e^{y}}$

$\Rightarrow \int y e^{y} dy = \int c \, dx$

$\Rightarrow y e^{y} - \int e^{y} dy = cx + d.$

$\Rightarrow y e^{y} - e^{y} = cx + d.$

$\Rightarrow e^{y}(y-1) = cx + d.$

Ans

$$\Rightarrow y'' + \frac{y'}{x} + \left(1 - \frac{1}{4x^2}\right)y = 0 \quad \text{— (1)}$$

Ans

Here $P = 1/x \Rightarrow -\int P\,dx = -\ln x$.

$$\therefore v = \frac{e^{-\int P dx}}{y_1^2} = \frac{e^{-\ln x}}{x^{-1}\cos^2 x} = \sec^2 x.$$

So $u = \int v\,dx = \int \sec^2 x\,dx = \tan x$.

As $y_2 = u y_1$

$$\Rightarrow y_2 = (\tan x)(x^{-1/2}\cos x) = x^{-1/2}\sin x.$$

Ans

(12) $(1-x^2)y'' - 2xy' + 2y = 0$, $y_1 = x$.

$$\Rightarrow y'' - \frac{2x}{(1-x^2)}y' + \frac{2}{(1-x^2)}y = 0 \quad \text{— (1)}$$

here $P = \frac{-2x}{(1-x^2)}$ & $\int P\,dx = \int \frac{2x}{x^2-1}dx = \int \frac{\frac{d}{dx}(x^2-1)}{x^2-1}dx$

$$\boxed{\Rightarrow \int P\,dx = \ln(x^2-1)}$$

$$\therefore v = \frac{e^{-\int P dx}}{y_1^2} = \frac{e^{-\ln(x^2-1)}}{x^2} = \frac{(x^2-1)^{-1}}{x^2}.$$

$$u = \int v\,dx = \int \frac{dx}{x^2(x^2-1)} = \int \frac{dx}{x^2(x+1)(x-1)}$$

$$\Rightarrow \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{x} + \frac{D}{x^2} = \frac{1}{x^2(x+1)(x-1)} \quad \text{— (2)}$$

$$\Rightarrow x^2(x-1)A + x^2(x+1)B + x(x+1)(x-1)C + (x+1)(x-1)D = 1$$

put $x - 1 = 0 \Rightarrow x = 1$.

$\Rightarrow 2B = 1 \Rightarrow B = 1/2$, Putting $x + 1 = 0 \Rightarrow x = -1$

$\Rightarrow -2A = 1 \Rightarrow A = -1/2$.

putting $x = 0$

$\Rightarrow D = -1$. & comparing coefficients of x^3 on b/s.

$\Rightarrow A + B + C = 0 \Rightarrow C = 1/2 - 1/2 + 0$

$\Rightarrow C = 0$.

$$\therefore \int \frac{dx}{x^2(x+1)(x-1)} = \int \frac{-dx}{2(x+1)} + \int \frac{dx}{2(x-1)} - \int \frac{dx}{x^2} =$$

$$= -\frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x-1) + \frac{1}{x}$$

$\Rightarrow$

Q5 $\frac{ds}{dt} \cdot \frac{d^2s}{dt^2} = const = 1$

Also $S(0) = 2$ m and $V(0) = 2$ m/s.

Find V & S at t = 6sec.

sol

$$\frac{ds}{dt} \cdot \frac{d^2s}{dt^2} = 1 \quad \text{—— (1)}$$

Let $z = \frac{ds}{dt} \Rightarrow z' = \frac{d^2s}{dt^2}$ putting in (1)

$\Rightarrow z z' = 1 \Rightarrow z\frac{dz}{dt} = 1$

$\Rightarrow \int z\,dz = \int dt \Rightarrow \frac{z^2}{2} = t + c$

$\Rightarrow z^2 = 2t + c. \Rightarrow \frac{ds}{dt} = (2t+c)^{1/2}.$

for $\frac{ds}{dt}(0) = 2. \Rightarrow 2 = c^{1/2} \Rightarrow c = 4.$

$$\therefore \frac{ds}{dt} = (2t+4)^{1/2} \quad \text{—— (2)}$$

$$\Rightarrow V = \frac{ds}{dt} = (2t+4)^{1/2}$$

$$\Rightarrow V(6) = (16)^{1/2} = 4.$$

(2) $\Rightarrow \frac{ds}{dt} = (2t+4)^{1/2} \Rightarrow \int ds = \int (2t+4)^{1/2} dt.$

$\Rightarrow S = \frac{(2t+4)^{1/2+1}}{2(1/2+1)} + d = \frac{(2t+4)^{3/2}}{3} + d.$

$\Rightarrow S(0) = \frac{(2(0)+4)^{3/2}}{3} + d$

$\Rightarrow 2 = \frac{2^3}{3} + d \Rightarrow d = -2/3.$

$\therefore S = \frac{(2t+4)^{3/2} - 2}{3}$

$S(6) = \frac{64-2}{3} = 62/3$ Ans

Q14 $\frac{d^2s}{dt^2} = \frac{ds}{dt}$.

$\Rightarrow z' = z$

$\Rightarrow \frac{dz}{dt} = z$

$\Rightarrow \ln z = t + c$

$\Rightarrow z = ce^{t}$.

$\Rightarrow \frac{ds}{dt} = ce^{t}$,

$\rightarrow \frac{ds}{dt}(0) = 2 = ce^{0} \Rightarrow c = 2$.

$\therefore \frac{ds}{dt} = 2e^{t}$ — (1)

$\frac{ds}{dt}(t=6) = 2e^{6}$.

Also

(1) $\Rightarrow \frac{ds}{dt} = 2e^{t}$.

integrating w.r.t on b/s.

$\Rightarrow \int ds = \int 2e^{t} dt$.

$\Rightarrow S = 2e^{t} + d$.

$\Rightarrow S(0) = 2 = 2e^{0} + d$.

$\Rightarrow 2 = 2 + d \Rightarrow d = 0$

$\therefore S = 2e^{t}$

$S(t=6) = 2e^{6}$.

Ans

Q Solve $y'' - y = 0$

$y'' - y = 0 \Rightarrow y'' = y \Rightarrow 2y'y'' = 2yy'$.

$\Rightarrow y'^2 = y^2 \Rightarrow y' = \pm y \Rightarrow y' = +y$,

$y' = -y$

$\Rightarrow \ln y = c + x$, $\ln y = c - x$

$\Rightarrow y = e^{x+c}$, $\ln y = e^{-x}$

$\Rightarrow y = ce^{x}$, $y = ce^{-x}$.

$\therefore y = y_1 + y_2 = ce^{x} + ce^{-x}$

Q15 $y'' = 2y'$. — (1)

$\Rightarrow$ Let $y' = z \Rightarrow y'' = z\frac{dz}{dy}$ putting in (1)

$\Rightarrow z\frac{dz}{dy} = 2z \Rightarrow z = 2y + c$

$\Rightarrow y' = 2y + c$ $\quad \because z = y'$

$\Rightarrow 1 = 2(0) + c$ $\quad \because y'_{(0,0)} = 1$

$$\Rightarrow \frac{1}{2}\ln(2y+1) = x + d.$$

$$\Rightarrow \ln(2y+1) = 2x + d \Rightarrow 2y = e^{2x+d} - 1$$

$$\Rightarrow y = (de^{2x} - 1)/2.$$

Now through the origin.

$$0 = \frac{(de^0 - 1)}{2} \Rightarrow d - 1 = 0 \Rightarrow d = 1.$$

$\therefore$ $y = (e^{2x} - 1)/2$ is the required curve.

INITIAL VALUE PROBLEMS

Verify that the given functions form a basis of the solutions of the given equations and solve given initial value problem.

(17) $y'' + 9y = 0$ — ① , $y(0) = 4$, $y'(0) = -6$,
$y_1 = \cos 3x$, $y_2 = \sin 3x$

sol
Let
$y_1 = \cos 3x$, $y_2 = \sin 3x$.
y_1 and y_2 form a basis if they are linearly independent.

$$\Rightarrow cy_1 + dy_2 = 0$$

$$\Rightarrow cy_1 = -dy_2 \Rightarrow y_1/y_2 = -d/c.$$

but $y_1/y_2 = \cot 3x \neq$ constant.
hence they are linearly independent and thus form the basis.

Now the general sol of ① is given by.

$$y = cy_1 + dy_2$$

$$\Rightarrow y = c\cos 3x + d\sin 3x \quad \text{— ②}$$

$$\Rightarrow 4 = c\cos 0 + d\sin 0 \qquad \because y(0) = 4.$$

$$\Rightarrow c = 4.$$

Also $y' = -3c\sin 3x + 3d\cos 3x.$

$\Rightarrow -6 = -3c\sin 0 + 3d\cos 0 \quad (\because y'(0) = -6)$

$\Rightarrow -6 = 3d\cos 0 \Rightarrow d = -2$

putting in ② we get the particular solution as

$\Rightarrow y = 4\cos 3x - 2\sin 3x$ Ans

⑱ $y'' + 2y' + y = 0$ —①, $y(0) = 1$, $y'(0) = 0$, e^{-x}, xe^{-x}

Sol Let $y_1 = e^{-x}$ & $y_2 = xe^{-x}$

y_1 and y_2 form a basis if they are linearly independent i-e they are not proportional,

i.e $y_1/y_2 \neq$ const

but $y_1/y_2 = \frac{e^{-x}}{xe^{-x}} = 1/x \neq$ const. hence they form the basis of solutions of given equations.

Now the general solution of —① is given by

$y = cy_1 + dy_2$.

$\Rightarrow y = ce^{-x} + dxe^{-x}$.

$\Rightarrow 1 = ce^0 + d(0) \quad (\because y(0) = 1)$.

$\Rightarrow c = 1$

Also $y' = -ce^{-x} + d(-xe^{-x} + e^{-x})$.

$\Rightarrow (0) = -ce^0 + d(0 + e^0) \Rightarrow 0 = -1 + d$.

$\Rightarrow d = 1$.

$\therefore$ the sol is.

$y = e^{-x} + xe^{-x}$. Ans

⑲ $4x^2y'' - 3y = 0$ —①, $y(1) = 3$, $y'(1) = 2.5$

$x^{-1/2}$, $x^{3/2}$

As $y_1/y_2 = \frac{x^{-1/2}}{x^{3/2}} = x^{-2} \neq$ const hence the solutions are linearly independent and thus form a basis.

Now the general solution of ① is given by.

$y = cx^{-1/2} + dx^{3/2} \Rightarrow 3 = c(1) + d(1) \Rightarrow 3 = c + d$ —②

Also $y' = -\frac{1}{2}cx^{-3/2} + \frac{3}{2}dx^{1/2} \Rightarrow 2.5 = -\frac{1}{2}c + \frac{3}{2}d \Rightarrow 5 = -c + 3d$ —③

$\Rightarrow 4d = 8 \Rightarrow d = 2$. putting in ② or ①

$\Rightarrow 5 = -c + 3(2) \Rightarrow c = 6 - 5 = 1$.

$\therefore y = 1x^{-1/2} + 2x^{3/2}$.

Ans

$$y'' - \frac{2x}{(1-x^2)} y' + \frac{2}{(1-x^2)} y = 0 \text{ --- ① } \quad y_1 = x.$$

Here $P = \frac{-2x}{(1-x^2)} \Rightarrow \int P dx = \ln|1-x^2|$

So $$v = \frac{e^{-\ln|1-x^2|}}{(x)^2} = \frac{(1-x^2)^{-1}}{x^2} = \frac{1}{x^2(1-x)(1+x)}.$$

Now
$$u = \int v\, dx = \int \frac{1}{x^2(1-x)(1+x)} dx$$

we solve this integral by partial fractions

$$1 = x^2(1-x)A + x^2(1+x)B + x(1-x)(1+x)C + (1-x)(1+x)D \text{ --- ②}$$

putting $1-x=0 \Rightarrow x = 1$ in ②

$\Rightarrow B = 1/2$

putting $1+x=0 \Rightarrow x = -1$ in ②

$\Rightarrow A = +1/2$.

putting $x = 0$ in ②

$\Rightarrow D = 1$

Comparing coefficients of x^3 on b/s of ②

$\Rightarrow -A + B - C = 0$

$\Rightarrow -C = -1/2 + 1/2 = 0$

So
$$\int \frac{dx}{x^2(1-x)(1+x)} = \int \frac{dx}{2(1+x)} + \int \frac{dx}{2(1-x)} - \int \frac{0\,dx}{x} + \int \frac{dx}{x^2}.$$

$\Rightarrow u = \frac{1}{2}\ln|1+x| + \frac{1}{2}\ln|1-x| - \cancel{\ln x} - 1/x$

$\Rightarrow u = \frac{1}{2}\left(\ln|1+x| + \ln|1-x|\right) - \cancel{\ln x} - \frac{1}{x}$

$\ln|(1+x)(1-x)| - \ln x - 1/x$

$$\Rightarrow u = \frac{1}{2}\ln\left|\frac{1-x^2}{x^2}\right| - \frac{1}{x}$$

$$\Rightarrow u = \frac{1}{2}\ln\left(\frac{1}{x^2}-1\right) - \frac{1}{x}$$

As $\quad y_2 = y_1(u) = x\,\frac{1}{2}\ln\left(\frac{1}{x^2}-1\right) - 1$

$$\Rightarrow y_2 = \frac{x}{2}\ln\left(\frac{1}{x^2}-1\right) - 1$$

<u>check</u> $\quad y_2' = \frac{1}{2}\left(x\left(\frac{x^2}{1-x^2}\right)\cdot\frac{d}{dx}\left(\frac{1}{x^2}-1\right) + \ln\left(\frac{1}{x^2}-1\right)\right)$

$$\Rightarrow y_2' = \frac{1}{2}\left(\frac{x^3}{(1-x^2)}\cdot\frac{-2}{x^3}\right) + \frac{1}{2}\ln\left(\frac{1}{x^2}-1\right)$$

$$\Rightarrow y_2' = \frac{-1}{(1-x^2)} + \frac{1}{2}\ln\left(\frac{1}{x^2}-1\right)$$

$$\Rightarrow y_2'' = \frac{0-(-1)(-2x)}{(1-x^2)^2} + \frac{1}{2}\left(\frac{x^2}{1-x^2}\right)\left(\frac{-2}{x^3}\right)$$

$$\Rightarrow y_2'' = \frac{-2x}{(1-x^2)^2} - \frac{1}{x(1-x^2)} \quad \text{putting in (1)}$$

$$\Rightarrow \frac{(-2x)}{(1-x^2)}(1/x^2) - 2x\left(-\frac{1}{(1-x^2)} + \frac{1}{2}\ln\left(\frac{1-x^2}{x^2}\right)\right) + 2\left(\frac{x}{2}\ln\left(\frac{1-x^2}{x^2}\right) - 1\right)$$

$$\Rightarrow \frac{-2x}{(1-x^2)} + \frac{2x}{(1-x^2)} - \frac{2x}{2}\ln\left(\frac{1-x^2}{x^2}\right) + \frac{2x}{2}\ln\left(\frac{1-x^2}{x^2}\right) - 2.$$

$$y_2 = \frac{x}{2}\ln(1-x^2) - 1$$

$$\Rightarrow y_2' = \frac{x}{2}\,\frac{-2x}{(1-x^2)} + \frac{1}{2}\ln(1-x^2) = \frac{-x^2}{(1-x^2)} + \frac{1}{2}\ln(1-x^2).$$

$$\Rightarrow y_2'' = -\frac{(2x(1-x^2) - x^2(-2x))}{(1-x^2)^2} + \frac{1}{2}\left(\frac{-2x}{1-x^2}\right)$$

$$\Rightarrow y_2'' = \frac{-(2x-2x^3+2x^3)}{(1-x^2)^2} - \frac{x}{(1-x^2)}$$

$$\Rightarrow y_2'' = \frac{-2x}{(1-x^2)^2} - \frac{x}{1-x^2} = \frac{-2x - x(1-x^2)}{(1-x^2)^2} = \frac{-2x-x+x^3}{(1-x^2)^2}$$

$$\Rightarrow \frac{x^3-3x}{(1-x^2)^2}$$

$u = \frac{1}{2}\ln(1-x^2) - \frac{1}{x}$ $\quad -2x(1-x^2) = -2x+2x^3$

$\Rightarrow y_2 = uy_1 = \frac{x}{2}\ln(1-x^2) - \frac{x}{x}$

$\Rightarrow y_2 = \frac{x}{2}\ln(1-x^2) - 1$

Check $\quad y_2' = \frac{1}{2}\left(x\frac{-2x}{(1-x^2)} + \ln(1-x^2)\right)$

$\Rightarrow y_2' = \frac{-x^2}{(1-x^2)} + \frac{1}{2}\ln(1-x^2)$

$\Rightarrow y_2'' = \frac{-2x(1-x^2)-(-x^2)(-2x)}{(1-x^2)^2} + \left(\frac{-x}{1-x^2}\right)$

$\Rightarrow y_2'' = \frac{-2x+2x^3-2x^3}{(1-x^2)^2} - \frac{x}{(1-x^2)}$

$\Rightarrow y_2'' = \frac{-2x}{(1-x^2)^2} - \frac{x}{(1-x^2)} = \frac{-2x-x+x^3}{(1-x^2)^2} = \frac{x^3-3x}{(1-x^2)^2}$

putting in ①

$\Rightarrow \frac{x^3-3x}{(1-x^2)^2}(1-x^2) - 2x\left(\frac{-x^2}{1-x^2} + \frac{\ln}{2}(1-x^2)\right) + 2\left(\frac{x}{2}\ln(1-x^2)-1\right)$

$\Rightarrow \frac{x^3-3x}{(1-x^2)} + \frac{2x^3}{(1-x^2)} - \frac{2x}{2}\ln(1-x^2) + x\ln(1-x^2) - 2$

$= \frac{2x^3}{(1-x^2)} - \frac{4x}{(1-x^2)}$

$\frac{x^3-3x}{(1-x^2)^2}(1-x^2) - 2x\left(\frac{-x^2}{1-x^2}\right) + \frac{1}{2}\ln(1-x^2) + 2\left(\frac{x}{2}\ln(1-x^2)-1\right)$

$= \frac{x^3}{(1-x^2)} - \frac{3x}{(1-x^2)} + \frac{2x^3}{(1-x^2)} - x\ln(1-x^2) + x\ln(1-x^2) - 2$

$= \frac{3x^3}{1-x^2} - \frac{3x}{1-x^2} - 2 = \frac{3x^3-3x-2(1-x^2)}{(1-x^2)}$

$= 3x^3 - 3x - 2 + 2x^2$