DIFFERENTIAL EQUATIONS

EXERCISE 1.8

Problems solved by;

Umair Asghar

NWFP, UET Peshawar

Differential Eqs are used for finding curves that intersect given curves at right angles, A task that arises rather often in practice. The new curves are then called the orthogonal trajectories of the given curves

Remember that angle of intersection of two curves is defined to be the angle b/w the tangents of the curves at point of intersection.

How To Find ORTHOGONAL TRAJECTORIES

Step # 01: We find a differential equation

$$y' = f(x,y). \quad \text{——} (1)$$

for which the given curves are solution curves.

Step #02: we write down the diff equations of the orthogonal trajectories to be found, which is.

$$y' = -\frac{1}{f(x,y)}. \quad \text{——} (2)$$

$\because$ Two lines are $\perp$ iff $m_1 m_2 = -1 \Rightarrow m_1 = -\frac{1}{m_2}$.

Step #03: Solve the differential equation (2)

Note that the differential eq (1) has infinitely many solutions curves, one for each value of the arbitrary constant c in its general solution. Hence we can write this family of curves as

$$F(x,y,c) = 0 \quad \text{——} (3)$$

e.g, a family of circles

$x^2+y^2=c$ can be written as.

$$F(x,y,c) = x^2+y^2-c=0$$

This is called a one-parameter family of curves and c is called the parameter of the family.

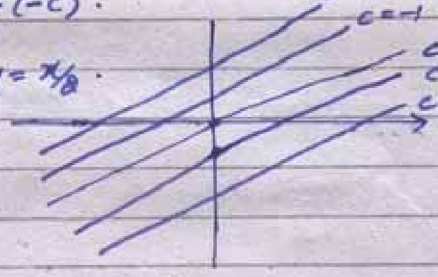
$\Rightarrow y = \frac{1}{2}x + (-c)$. Required diff. Equation.

Q4

for $c = 0$, $y = x/2$.

for $c = 1$

$y = \frac{1}{2}x - 1$



Q5 $y = ce^{2x} \Rightarrow c = y/e^{2x}$.

$y' = 2ce^{2x} = 2\left(\frac{y}{e^{2x}}\right)e^{2x} = 2y$.

$\Rightarrow y' = 2y$. Required differential equatione.

Q6 $y = \tan(x+c) \Rightarrow c = \tan^{-1}y - x = c$

$\Rightarrow y' = \sec^2(x+c)$ –

$\Rightarrow y' = \sec^2(\cancel{x} + (\tan^{-1}(y) - \cancel{x}))$.

Required differential equation.

Q8. $y = cx^4$

$\Rightarrow y' = 4cx^3 = 4\left(\frac{y}{x^4}\right)(x^3) = \frac{4y}{x}$ $\Rightarrow c = y/x^4$.

Q9. $c^2x^2 + y^2 = c^2$

$\Rightarrow c^2x^2 - c^2 = -y^2 \Rightarrow c^2(x^2-1) = -y^2$.

$\Rightarrow c^2 = -\frac{y^2}{x^2-1} \Rightarrow c^2 = \frac{y^2}{1-x^2}$.

$c^2x^2 + y^2 = c^2$ diff w.r.t x.

$\Rightarrow 2c^2x + 2yy' = 0 \Rightarrow y' = -\frac{2c^2x}{2y} = -\frac{c^2x}{y}$.

putting values of c^2

$\Rightarrow y' = -\left(\frac{y^2}{1-x^2}\right)\left(\frac{x}{y}\right) = -\frac{xy}{(1-x^2)} = \frac{xy}{x^2-1}$

Ans

ORTHOGONAL TRAJECTORIES

Find the orthogonal Trajectories. Plot and sketch some curves and Trajectories.

Q10 $y = ce^{x^2}$.

$\Rightarrow y' = 2cxe^{x^2}$. slope of the given curve.

$y' = -\frac{1}{2cxe^{x^2}}$; putting $c = y/e^{x^2} \Rightarrow y' = -\frac{1}{2yx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2xy}$

$\Rightarrow \int 2y\,dy = \int -\frac{1}{x}\,dx$.

$\Rightarrow y^2 = -\ln x + \ln c^* \Rightarrow y^2 = \ln c^*/x$.

$\Rightarrow e^{y^2} = c^*/x \Rightarrow xe^{y^2} = c^*$

Ans

(11) $y = ce^{-x} \Rightarrow c = y/e^{-x}$.

$\Rightarrow y' = -ce^{-x} \Rightarrow y' = -(y/e^{-x})(e^{-x}) \Rightarrow y' = -y$.

hence the slope of the line Tangent to the curve which is perpendicular to the line tangent to the given curve is.

$y' = -\left(\frac{-1}{y}\right) \Rightarrow y' = \frac{1}{y}$.

$\Rightarrow \int y\,dy = \int dx \Rightarrow \frac{y^2}{2} = x + \ln c^*$

$\Rightarrow y^2 = 2x + c^*$. Ans. $\Rightarrow y = \sqrt{2x + c^*}$ Ans

(12) $y = \ln|x| + c \Rightarrow c = y - \ln x$.

$\Rightarrow y' = \frac{1}{x} + 0 \Rightarrow y' = \frac{1}{x}$.

hence the slope of the Tangent to the curve which is $\perp$ to the given curve is.

$y' = -\frac{1}{1/x} \Rightarrow y' = -x$.

$\therefore \int dy = -\int x\,dx$.

$\Rightarrow y = -\frac{x^2}{2} + c^*$. Ans.

Q13 slope of the given curve.

Now the slope of the curve ⊥ to the given curve is.

$$y' = -\frac{1}{4/2x} \Rightarrow y' = -2x/y$$

and thus the required solution could be obtained by

$$\int y\,dy = -2\int x\,dx \Rightarrow \frac{y^2}{2} = -x^2 + c^*$$

Ans

Q14 $y = \sqrt{x+c}$

$$\Rightarrow y' = \frac{1}{2\sqrt{x+c}} \Rightarrow y' = \frac{1}{2\sqrt{x + y^2 - x}} \quad \because c = y^2 - x.$$

$\Rightarrow y' = \frac{1}{2y}$. slope of the line tangent to the given curve.

Now the slope of the curve ⊥ to given curve is.

$$y' = -\frac{1}{1/2y} = -2y$$

$$\Rightarrow y' = -2y \Rightarrow \int \frac{dy}{y} = \int -2\,dx.$$

$$\Rightarrow \ln y = -2x + c^* \Rightarrow y = ce^{-2x}.$$ Ans

Q15 $y = cx^{3/2}$

$$\Rightarrow y' = \frac{3}{2}cx^{1/2} \Rightarrow y' = \frac{3}{2}\left(\frac{y}{x^{3/2}}\right)(x^{1/2}).$$

$\Rightarrow y' = \frac{3}{2}(y/x)$ slope of the given curve.

Hence the slope of the curve ⊥ to the given curve is.

$$y' = \frac{-1}{3/2\,(y/x)} = -\frac{2}{3}(x/y).$$

$$\Rightarrow \int 3y\,dy = \int -2x\,dx \Rightarrow \frac{3}{2}y^2 = -x^2 + c^*$$

$$\Rightarrow 3y^2 = -2x^2 + c^*$$

Ans

this represents the slope of the given curve. And the slope of the curve $\perp$ to the given curve is.

$$y' = -\left(-\frac{1}{y/x}\right) = x/y.$$

$$\Rightarrow \int y\,dy = \int x\,dx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + \frac{c}{2}$$

$$\Rightarrow x^2 + c = y^2 \quad \text{Ans}$$

Q#17 $y = c/x^2$.

$$\Rightarrow y' = -\frac{2c}{x^3} = -\frac{2(yx^2)}{x^3} = -2y/x.$$

slope of the given curve.

hence the slope of the curve $\perp$ to the given curve is.

$$y' = -\left(-\frac{1}{2y/x}\right) = +\frac{x}{2y}.$$

$$\Rightarrow \int 2y\,dy = \int x\,dx \Rightarrow y^2 = \frac{x^2}{2} + c^*. \quad \text{Ans}$$

Q#18 $(x-c)^2 + y^2 = c^2$. — ①

$$\Rightarrow 2(x-c) + 2yy' = 0.$$

$$\Rightarrow y' = -\frac{2(x-c)}{2y} = -\frac{(x-c)}{y} \quad \text{— ②}$$

from ① $\cancel{c^2} + x^2 - 2cx + y^2 = \cancel{c^2}$

$$\Rightarrow \frac{x^2+y^2}{2x} = c \quad \text{putting in ②}$$

$$\Rightarrow y' = -\frac{\left(x - \frac{x^2+y^2}{2x}\right)}{y} = -\frac{(2x^2 - x^2 - y^2)}{2xy}$$

$$\Rightarrow y' = -\left(\frac{x^2 - y^2}{2xy}\right) \Rightarrow y' = \frac{y}{2x} - \frac{x}{2y} \quad \text{— ③}$$

slope of the given curve.

hence the slope of the curve $\perp$ to the given curve is.

$$y' = -\left(\frac{1}{\frac{y}{2x} - x/2y}\right) \Rightarrow y' = -\left(\frac{1}{\frac{2y^2 - 2x^2}{4xy}}\right)$$

$$\Rightarrow y' = \frac{4xy}{2x^2 - 2y^2} \Rightarrow y' = \frac{2xy}{x^2 - y^2}.$$

$\Rightarrow \frac{1}{y'} = \frac{x^2}{2xy} - \frac{y^2}{2xy} = \frac{x}{2y} - \frac{y}{2x}$.

$\Rightarrow \frac{dx}{dy} = \frac{1}{2}\left(x/y - y/x\right)$.

Let $y/x = u$ ~~$\Rightarrow x = uy$~~

$\Rightarrow y = ux \Rightarrow y' = u'x + u$

$\Rightarrow \frac{1}{y'} = \frac{1}{xu'+u}$

$\frac{1}{xu'+u} = \frac{1}{2}\left(1/u - u\right) \Rightarrow \frac{1}{xu'+u} = \frac{1}{2}\left(\frac{1-u^2}{u}\right)$.

$\Rightarrow \frac{2u}{1-u^2} = xu' + u$.

$\Rightarrow \frac{2u}{1-u^2} - u = xu' \Rightarrow \frac{2u - u(1-u^2)}{1-u^2} = xu'$

$\Rightarrow \frac{2u - u + u^3}{1-u^2} = xu'$

$y' = \frac{2xy}{x^2-y^2}$

$\Rightarrow x' = \frac{x^2-y^2}{2xy} = \frac{x}{2y} - \frac{y}{2x}$.

Now let $u = x/y \Rightarrow x = uy \Rightarrow x' = u'y + u$

$\Rightarrow u'y + u = \frac{u}{2} - \frac{1}{2u}$

$\Rightarrow u'y = -\frac{u}{2} - \frac{1}{2u} \Rightarrow u'y = -\frac{1}{2}\left(u - \frac{1}{u}\right) = -\frac{1}{2}\left(\frac{u^2-1}{-u}\right)$

$\Rightarrow \frac{uu'}{(u^2-1)} = -\frac{1}{2y} \Rightarrow \int \frac{u\,du}{(u^2-1)} = -\frac{1}{2}\int \frac{dy}{y}$

$\Rightarrow \frac{1}{2}\ln(u^2-1) = -\frac{1}{2}\ln y + \ln c$

$\Rightarrow \frac{(u^2-1)^{1/2}}{c} = y^{-1/2} \Rightarrow (u^2-1) = c^2/y$.

$\Rightarrow (x^2/y^2 - 1) = c^2/y \Rightarrow x^2 - y^2 = \frac{c^2y^2}{y}$

$\Rightarrow x^2 - y^2 = c^2 y$.

$\Rightarrow x^2 - c^2 y - y^2 = 0$.