DIFFERENTIAL EQUATIONS

EXERCISE 1.5

Problems solved by;

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$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$

$\Rightarrow du = 2x\,dx + 8y\,dy = 0$

for $du = 0$

$\Rightarrow 2x\,dx + 8y\,dy = 0$ Ans

(2) $u = x^2 - y^2$

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$

$\Rightarrow du = 2x\,dx + (-2y)dy = 0$

$\Rightarrow 2x\,dx - 2y\,dy = 0$ Ans

(3) $u = e^{x^2/y}$

$\Rightarrow du = e^{x^2/y}(2x/y)dx + e^{x^2/y}(-x^2/y^2)dy = 0$

$\Rightarrow \frac{2x}{y}e^{x^2/y}dx - \frac{x^2}{y^2}e^{x^2/y}dy = 0$ Ans

(4) $u = \frac{1}{(x^2+y^2)}$

$\Rightarrow du = -1(x^2+y^2)^{-2}\cdot 2x\,dx + (-1(x^2+y^2)^{-2})\cdot 2y\,dy = 0$

$\Rightarrow du = \frac{-2x}{(x^2+y^2)^2}dx - \frac{2y}{(x^2+y^2)^2}dy = 0$

$\Rightarrow \frac{-2x}{(x^2+y^2)^2}dx - \frac{2y\,dy}{(x^2+y^2)^2} = 0$ Ans

(5) $u = \tan(y^2 - x^2)$

$\Rightarrow du = \sec^2(y^2-x^2)(-2x)\,dx + 2y\sec^2(y^2-x^2)\,dy = 0$

$\Rightarrow -2x\sec^2(y^2-x^2)dx + 2y\sec^2(y^2-x^2)dy = 0$ Ans

(6) $u = \sin x \cosh y$

$\Rightarrow du = \cos x \cosh y\,dx + \sin x \sinh y\,dy = 0$

$\Rightarrow \cos x \cosh y\,dx + \sin x \sinh y\,dy = 0$

ons are exact and solve them

(1) $2xy\,dx + x^2dy = 0$

Solution

Let $M = 2xy$ & $N = x^2$.

Check for Exactness.

$\Rightarrow \frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} = 2x$

Here this is an exact differential Eq

Now for solution.

$u(x,y) = \int M\,dx + k(y)$.

$\Rightarrow u = \int 2xy\,dx + k(y)$

$\Rightarrow u = x^2y + k(y)$ —— (1)

differentiating w.r.t y.

$\Rightarrow \frac{\partial u}{\partial y} = x^2 + \frac{dk}{dy}$

but $\frac{\partial u}{\partial y} = N = x^2$.

$\therefore \frac{dk}{dy} = 0 \Rightarrow k = \text{constant} = c$

Now the general sol is putting in (1)

$\Rightarrow u = x^2y = c$.

$\Rightarrow x^2y = c$

Ans

(2) $-yx^{-2}dx + x^{-1}dy = 0$

Solution

Let $M = -yx^{-2}$ and $N = x^{-1}$.

$\frac{\partial M}{\partial y} = -x^{-2} = \frac{\partial N}{\partial x} = -x^{-2}$.

Here the equation is exact.

Now for solution.

$u = \int M\,dx + k(y)$

$\Rightarrow u = \int -yx^{-2}dx + k(y)$

$\Rightarrow u = \frac{-yx^{-1}}{(-2+1)} + k(y)$

Now differentiating with respect to y.

$\Rightarrow \frac{\partial u}{\partial y} = +\frac{x^{-1}}{1} + \frac{dk}{dy}$.

but

$\frac{\partial u}{\partial y} = N = x^{-1} + \frac{dk}{dy}$.

$\Rightarrow \cdots + \frac{dk}{dy} \quad x^{-1} \Rightarrow k = \text{const}$

$M \qquad N$

$\Rightarrow \frac{\partial M}{\partial \theta} = 3e^{3\theta} = \frac{\partial N}{\partial r} = 3e^{3\theta}$

Hence the eq is exact.

Now for the sol

$u = \int M dr$

$\Rightarrow u = \int e^{3\theta} dr$

$\Rightarrow u = e^{3\theta} \cdot r + k(\theta)$ —— ①

$\Rightarrow \frac{\partial u}{\partial \theta} = 3e^{3\theta} \cdot r + \frac{dk}{d\theta}$

$\Rightarrow \frac{\partial u}{\partial \theta} = N = e^{3\theta} \cancel{3r} = 3e^{3\theta} \cdot r + \frac{dk}{d\theta}$

$\Rightarrow k = \text{const}$

Hence the general solution is

$u(x,y) = re^{3\theta} = c = \text{const}$.

Ans

⑪ $e^{-2\theta}(r dr - r^2 d\theta) = 0$

$\Rightarrow re^{-2\theta} dr - r^2 e^{-2\theta} d\theta = 0$

Let $M = re^{-2\theta}$ & $N = -r^2 e^{-2\theta}$

$\Rightarrow \frac{\partial M}{\partial \theta} = -2re^{-2\theta} = \frac{\partial N}{\partial r} = -2re^{-2\theta}$

So the equation is exact.

Now we have

$u = \int N d\theta + k(x)$

$\Rightarrow u = \int -r^2 e^{-2\theta} d\theta + k(x)$

$\Rightarrow u = \frac{-r^2 e^{-2\theta}}{-2} + k(x)$

$\Rightarrow u = \frac{r^2 e^{-2\theta}}{2} + k(x)$ —— ①

$\Rightarrow \frac{\partial u}{\partial r} = \frac{2re^{-2\theta}}{2} + k(x)$

$\Rightarrow \frac{\partial u}{\partial r} = re^{-2\theta} + k(x)$ —— ②

but $\frac{\partial u}{\partial r} = M = re^{-2\theta}$. putting in ②

$\Rightarrow re^{-2\theta} = re^{-2\theta} + \frac{dk(x)}{dx}$

$\Rightarrow k = \text{const}$.

$u(x,y) = \frac{r^2 e^{-2\theta}}{2} = c = \text{const}$.

$\Rightarrow r^2 e^{-2\theta} = 2c$

$\Rightarrow r^2 e^{-2\theta} = C$ Ans

⑫ $(\cot y + x^2) dx = x \operatorname{cosec}^2 y \, dy$.

$\Rightarrow (\cot y + x^2) dx - x \operatorname{cosec}^2 y \, dy = 0$

Let $M = \cot y + x^2$ & $N = -x \operatorname{cosec}^2 y$

$\Rightarrow \frac{\partial M}{\partial y} = -\operatorname{cosec}^2 y = \frac{\partial N}{\partial x} = -\operatorname{cosec}^2 y$.

Hence the equation is exact.

Now we have

$u = \int M dx$.

$\Rightarrow u = \int (\cot y + x^2) dx$.

$\Rightarrow u = x\cot y + \frac{x^3}{3} + k(y)$ —— ①

$\Rightarrow \frac{\partial u}{\partial y} = -x \operatorname{cosec}^2 y + \frac{dk}{dy}$ —— ②

but $\frac{\partial u}{\partial y} = N = -x \operatorname{cosec}^2 y$.

putting in ②

$\Rightarrow -x \operatorname{cosec}^2 y = -x \operatorname{cosec}^2 y + \frac{dk}{dy}$

$\Rightarrow \frac{dk}{dy} = 0$

$\Rightarrow k = \text{const}$, Inserting into ①

we get the general solution as

$u(x,y) = x\cot y + \frac{x^3}{3} = c = \text{const}$.

Ans

check differentiating the Ans w.r.t x

$\Rightarrow x(-\operatorname{cosec}^2 y) \frac{dy}{dx} + \cot y + x^2 = 0$

$\Rightarrow -x \operatorname{cosec}^2 y \frac{dy}{dx} + (\cot y + x^2) = 0$

$\Rightarrow (\cot y + x^2) dx = x \operatorname{cosec}^2 y \, dy$

hence correct

Let $M = 3y^2$ & $N = x$

$\Rightarrow \frac{\partial M}{\partial y} = 6y$ & $\frac{\partial N}{\partial x} = 1$.

Hence the equation is not exact. To make it exact we have to find the I.F. And we have the formula.

$I.F = \exp\int\left(\frac{My - Nx}{x}\right)dx$

$\Rightarrow I.F = \exp\int\left(\frac{6y-1}{x}\right)dx$.

$\Rightarrow I.F = \exp\left((6y-1)\int\frac{1}{x}dx\right)$

$= \exp((6y-1)\ln x)$

$\Rightarrow I.F = \exp(\ln x^{(6y-1)})$

$\Rightarrow I.F = x^{(6y-1)}$

$y(\pi/4) = 3.8$

Let $M = 2y^{-1}\cos 2x$

& $N = -y^{-2}\sin 2x$.

So that $\frac{\partial M}{\partial y} = -2y^{-2}\cos 2x$.

& $\frac{\partial N}{\partial x} = -2y^{-2}\cos 2x$.

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ hence the equation is exact.

Now we have

$u = \int M dx$.

$\Rightarrow u = \int 2y^{-1}\cos 2x\, dx$

$\Rightarrow u = 2y^{-1}\int\cos 2x\, dx$

$\Rightarrow u = 2y^{-1}\frac{\sin 2x}{2} + k(y)$.

$\Rightarrow u = y^{-1}\sin 2x + k(y)$ — (a)

differentiating w.r.t y

$\Rightarrow \frac{\partial u}{\partial y} = -y^{-2}\sin 2x + \frac{dk}{dy}$.

but $\frac{\partial u}{\partial y} = N = -y^{-2}\sin 2x$.

Hence $-y^{-2}\sin 2x = -y^{-2}\sin 2x + \frac{dk}{dy}$

$\Rightarrow \frac{dk}{dy} = 0$

$\Rightarrow k = \text{const}$. Inserting into (a) we get the solution as

$u(x,y) = y^{-1}\sin 2x = c$ — (b)

Now given that $y(\pi/4) = 3.8$

(b) $\Rightarrow (3.8)^{-1}\sin(2(\pi/4)) = c$

$\Rightarrow c = \frac{\sin(\pi/2)}{3.8}$

$\Rightarrow c = \frac{1}{3.8}$ putting in (b)

$\Rightarrow y^{-1}\sin 2x = \frac{1}{3.8}$

or $y = 3.8\sin 2x$

Ans

Let $M = x^2 + y^2$ & $N = -2xy$.

So that $\frac{\partial M}{\partial y} = 2y \neq \frac{\partial N}{\partial x} = -2y$.

Hence the equation is not exact. In order to make it exact, we have to find the I.F i.e.

$I.F = \exp\left[\int\left(\frac{My - Nx}{N}\right)dx\right]$.

$\Rightarrow I.F = \exp\left[\int\left(\frac{2y + 2y}{-2xy}\right)dx\right]$.

$\Rightarrow I.F = \exp\left[\int \frac{4y}{-2xy}\, dx\right]$.

$\Rightarrow I.F = \exp\left[-2\int \frac{1}{x}\, dx\right]$.

$\Rightarrow I.F = \exp\left[-2\ln x\right]$.

$\Rightarrow I.F = \exp \ln x^{-2}$.

$\Rightarrow I.F = x^{-2}$.

Multiplying with ⓐ

$\Rightarrow x^{-2}(x^2 + y^2)\cancel{dx} - 2xy \cdot x^{-2} dy = 0$

$\Rightarrow \left(1 + y^2/x^2\right) dx - 2y/x\, dy = 0$

Hence $\frac{\partial M}{\partial y} = 2y/x^2 = \frac{\partial N}{\partial x} = 2y/x^2$

Hence the eq. becomes exact.

Now we have -

$u = \int M dx + k(y)$

$\Rightarrow u = \int\left(1 + y^2/x^2\right)dx + k(y)$

$\Rightarrow u = x + \left(\frac{-y^2}{x}\right) + k(y)$.

$\Rightarrow u = x - \frac{y^2}{x} + k(y)$.

$\Rightarrow u = \frac{x^2 - y^2}{x} + k(y)$. — ⓑ

$\Rightarrow \frac{\partial u}{\partial y} = \frac{-2y}{x} + \frac{dk}{dy}$.

but $\frac{\partial u}{\partial y} = \frac{-2y}{x} = N$.

$\Rightarrow k = \text{const}$. inserting in ⓑ to get the sol.

$u(x, y) = \frac{x^2 - y^2}{x} + c$

or $\frac{x^2 - y^2}{x} = c$. — ⓒ

Now given that $y(1) = 2$

ⓒ $\Rightarrow \frac{1 - 4}{1} = c$

$\Rightarrow c = -3$. putting in ⓒ

$\Rightarrow x^2 - y^2 = -3x$.

$\Rightarrow x^2 - y^2 + 3x = 0$ Ans

(b) $ye^x dx + (2y + e^x) dy = 0$

$y(0) = -1$

Let $M = ye^x$ & $N = 2y + e^x$

so that $\frac{\partial M}{\partial y} = e^x$ and $\frac{\partial N}{\partial x} = e^x$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ hence the equation is exact

Now for solution.

$u = \int M dx$.

$\Rightarrow u = \int ye^x dx$.

$\Rightarrow u = ye^x + k(y)$ — ⓐ

$\Rightarrow \frac{\partial u}{\partial y} = e^x + \frac{dk}{dy}$.

but $\frac{\partial u}{\partial y} = N = 2y + e^x$.

hence $2y + \cancel{e^x} = \cancel{e^x} + \frac{dk}{dy}$

$\Rightarrow \int \frac{dk}{dy} dy = \int 2y\, dy$.

$\Rightarrow k = y^2 + c$ inserting in ⓐ we have the solution.

$\Rightarrow u = ye^x + y^2 = c$

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$\frac{\partial M}{\partial y} = -e^y = \frac{\partial N}{\partial x} = -e^y.$

Hence the equation is exact.

Now

$u = \int((x+1)e^x - e^y)\,dx.$

$\Rightarrow u = \int(xe^x + e^x - e^y)\,dx$

$\Rightarrow u = xe^x - \int e^x dx + e^x - xe^y + k(y)$

$\Rightarrow u = xe^x - e^x + e^x - xe^y + k(y)$

$\Rightarrow u = x(e^x - e^y) + k(y) \quad \text{--- (a)}$

$\Rightarrow \frac{\partial u}{\partial y} = -xe^y + \frac{dk}{dy}.$

but $\frac{\partial u}{\partial y} = N = -xe^y$

$\therefore \frac{dk}{dy} = +\cancel{xe^y} - \cancel{xe^y}$

$\Rightarrow \frac{dk}{dy} = 0 \Rightarrow k = \text{const}.$

putting in (a) to get the general solution as.

$u(x,y) = x(e^x - e^y) = \text{const} = c$

$\Rightarrow x(e^x - e^y) = c \quad \text{--- (b)}$

Now $y(1) = 0$.

(b) $\Rightarrow 1(e^1 - e^0) = c$

$\Rightarrow e - 1 = c$ putting in (b)

$\Rightarrow x(e^x - e^y) = e - 1$

Ans

(20) $(2xy\,dx + dy)e^{x^2} = 0,\ y(0)=2$

$\Rightarrow \underbrace{e^{x^2}2xy}_{M}\,dx + \underbrace{e^{x^2}}_{N}dy = 0$

so that

$\frac{\partial M}{\partial y} = 2xe^{x^2} = \frac{\partial N}{\partial x} = 2xe^{x^2}.$

Hence the eq. is exact.

$u = \int N\,dy$

$\Rightarrow u = \int e^{x^2} dy$

$\Rightarrow u = e^{x^2}y + k(x) \quad \text{--- (a)}$

$\Rightarrow \frac{\partial u}{\partial x} = 2xye^{x^2} + \frac{dk}{dx}$

but $\frac{\partial u}{\partial x} = M = 2xye^{x^2}$;

Hence

$\cancel{2xye^{x^2}} = \cancel{2xye^{x^2}} + \frac{dk}{dx}$

$\Rightarrow k = \text{const}$ putting in (a) to get.

$u(x,y) = e^{x^2}y + c.$

or $e^{x^2}y + c = 0 \quad \text{--- (b)}$

Now $y(0) = 2$.

$\Rightarrow e^0(2) + c = 0.$

$\Rightarrow 2 = -c \Rightarrow c = -2.$ putting in (b)

$\Rightarrow e^{x^2}y = 2$

Ans

$(ax+by)dx + (kx+ly)dy = 0$ exact? (here a, b, k, l are constant) Solve the exact eq.

Sol

$$\underbrace{(ax+by)}_{M}dx + \underbrace{(kx+ly)}_{N}dy = 0 \quad \text{— (A)}$$

$$\frac{\partial M}{\partial y} = b \quad \& \quad \frac{\partial N}{\partial x} = k$$

Hence the equation could only be exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow b = k$. putting in (A) to make it exact.

$$(ax+by)dx + (bx+ly)dy = 0$$

Now

$$u = \int (ax+by)\,dx$$

$$\Rightarrow u = \frac{ax^2}{2} + bxy + k(y) \quad \text{— (B)}$$

$$\Rightarrow \frac{\partial u}{\partial y} = bx + \frac{dk}{dy}$$

but $\frac{\partial u}{\partial y} = N = bx + ly$.

$$\therefore \cancel{bx} + ly = \cancel{bx} + \frac{dk}{dy}$$

$$\Rightarrow \frac{dk}{dy} = ly$$

$$\Rightarrow k = \frac{ly^2}{2} + c$$

putting in (B) to get the solution as

$$u(x,y) = \frac{ax^2}{2} + bxy + \frac{ly^2}{2} + c$$

or

$$\frac{ax^2}{2} + bxy + \frac{ly^2}{2} = c$$

Ans

... are I.F of $ydx + 2xdy$.

$$\underbrace{y}_{M}dx + \underbrace{2x}_{N}dy = 0 \quad \text{— (1)}$$

$$\frac{\partial M}{\partial y} = 1 \quad \& \quad \frac{\partial N}{\partial x} = 2$$

Hence the equation is not exact so lets check that whether y, xy^3 & x^2y^5 are I.F or not.

"y"

$$y^2dx + 2xy\,dy = 0$$

$$\frac{\partial M}{\partial y} = 2y \quad \& \quad \frac{\partial N}{\partial x} = 2y$$

Hence the equation becomes exact.

"xy^3"

$$xy^4dx + 2x^2y^3dx$$

$$\frac{\partial M}{\partial y} = 4xy^3 = \frac{\partial N}{\partial x} = 4xy^3$$

so exact

"x^2y^5"

$$x^2y^5(ydx) + x^2y^5(2x)dy = 0$$

$$\Rightarrow x^2y^6dx + 2x^3y^5dy = 0$$

$$\Rightarrow \frac{\partial M}{\partial y} = 6x^2y^5 = \frac{\partial N}{\partial x} = 6x^2y^5.$$

Exact Ans

multiplying with e^x.

$\Rightarrow e^x\sin y\,dx + e^x\cos y\,dy = 0$. ——(1)

$\Rightarrow$ let $M = e^x\sin y$ & $N = e^x\cos y$.

Q25 so that $\frac{\partial M}{\partial y} = e^x\cos y$ and $\frac{\partial N}{\partial x} = e^x\cos y$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. hence e^x is an I.F

or $I.F = e^{\int \frac{1}{N}(M_y - N_x)dx}$.

$\Rightarrow I.F = e^{\int \frac{1}{\cos y}(\cos y - 0)\,dx}$

$\Rightarrow I.F = e^x$

Now for solution of (1)

$\Rightarrow u = \int M\,dx \Rightarrow u = \int e^x\sin y\,dx \Rightarrow u = e^x\sin y + k(y)$ —(2)

$\Rightarrow \frac{\partial u}{\partial y} = e^x\cos y + \frac{dk}{dy}$ but $\frac{\partial u}{\partial y} = N = e^x\cos y$.

$\therefore e^x\cos y = e^x\cos y + \frac{dk}{dy} \Rightarrow k = \text{const}$.

putting in (2) to get the solution.

$u(x,y) = e^x\sin y = c = \text{const}$. Ans

(26) $y\,dx + [y + \tan(x+y)]dy = 0$, $\cos(x+y)$

multiplying with the term / function $\cos(x+y)$.

$\Rightarrow y\cos(x+y)dx + (y\cos(x+y) + \tan(x+y)\cos(x+y))dy =$

Let $M = y\cos(x+y)$ & $N = y\cos(x+y) + \tan(x+y)\cos(x+y)$

so that

$\frac{\partial M}{\partial y} = -y\sin(x+y) + \cos(x+y)$

$= \cos(x+y) - y\sin(x+y)$

& $\frac{\partial N}{\partial x} = -y\sin(x+y) + \sec^2(x+y)\cos(x+y) + \tan(x+y)(-\sin(x+y))$

$= -y\sin(x+y) + \sec(x+y) - \sin^2(x+y)\sec(x+y)$

$= -y\sin(x+y) + \sec(x+y)(1 - \sin^2(x+y))$.

$= -y\sin(x+y) + \sec(x+y)\cos^2(x+y)$.

$\Rightarrow \frac{\partial N}{\partial x} = -y\sin(x+y) + \cos(x+y)$.

since $\frac{\partial M}{\partial y} = \partial N/\partial x$ hence the equation is

$u = \int M\,dx \Rightarrow u = \int y\cos(x+y)\,dx$

$\Rightarrow u = +y\sin(x+y) + k(y)$. —— ①

differentiating w.r.t y. we get.

$\frac{\partial u}{\partial y} = \sin(x+y) + y\cos(x+y) + \frac{dk}{dy}$.

but $\frac{\partial u}{\partial y} = N = y\cos(x+y) + \cos(x+y)\frac{\sin(x+y)}{\cos(x+y)}$.

$= y\cos(x+y) + \sin(x+y)$.

$\Rightarrow y\cos(x+y) + \sin(x+y) = \sin(x+y) + y\cos(x+y) + \frac{dk}{dy}$

$\Rightarrow k = const$ putting in ① to get the solution as

$u(x+y) = y\sin(x+y) = const = c$.

Ans

check $du = M\,dx + N\,dy$.

$\Rightarrow du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = y\cos(x+y)dx + (y\cos(x+y) + \sin(x+y))dy$

verified

(b) $(a+1)y\,dx + (b+1)x\,dy = 0$ — ① $x^a y^b$

Here I.F $= x^a y^b$. multiplying with the eq ①

$\Rightarrow x^a y^b (a+1) y\,dx + x^a y^b (b+1) x\,dy = 0$

$\Rightarrow x^a y^{b+1}(a+1)dx + x^{a+1} y^b (b+1) dy = 0$.

Now let $M = x^a y^{b+1}(a+1)$ & $N = x^{a+1} y^b (b+1)$

So that $\frac{\partial M}{\partial y} = (a+1)(b+1)x^a y^b = \frac{\partial N}{\partial x} = (a+1)(b+1)x^a y^b$.

Now in order to get the solution solve.

$u = \int M\,dx \Rightarrow u = \int (a+1)x^a y^{b+1}\,dx$.

$\Rightarrow u = \frac{(a+1)(y^{b+1})(x^{a+1})}{a+1} + k(y)$.

$\Rightarrow u = (y^{b+1})(x^{a+1}) + k(y)$. —— ②

$\Rightarrow \frac{\partial u}{\partial y} = (b+1)(y^b)(x^{a+1}) + \frac{dk}{dy}$

but $\frac{\partial u}{\partial y} = N = (b+1)x^{a+1}y^b$.

$\Rightarrow (b+1)y^b(x^{a+1}) = (b+1)y^b(x^{a+1}) + \frac{dk}{dy}$

check

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\Rightarrow du = (a+1)x^{a}y^{b+1}dx + (b+1)y^{b}x^{a+1}.$$

(28) $3(y+1)dx = 2x\,dy$ —(1), $(y+1)/x^4$.

Here I.F $= (y+1)/x^4$ multiplying with —(1)

$$\Rightarrow \frac{3(y+1)^2}{x^4}dx - \frac{2(y+1)}{x^3}dy = 0$$

Let $M = \frac{3(y+1)^2}{x^4}$ & $N = -\frac{2(y+1)}{x^3}$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{6(y+1)}{x^4} = \frac{\partial N}{\partial x} = +\frac{6(y+1)}{x^4}$$

Hence the equation is exact now

In order to find its solution, we have,

$$u = \int M dx \Rightarrow u = \int \frac{3(y+1)^2}{x^4}dx$$

$$\Rightarrow u = -\frac{\cancel{3}(y+1)^2}{\cancel{3}x^3} + k(y). \quad —(2)$$

Now $\frac{\partial u}{\partial y} = -\frac{2(y+1)}{x^3} + \frac{dk}{dy}$

but $\frac{\partial u}{\partial y} = N = \cancel{-\frac{2(y+1)}{x^3}} = \cancel{-\frac{2(y+1)}{x^3}} + \frac{dk}{dy}$,

$$\Rightarrow \frac{dk}{dy} = 0 \Rightarrow k = \text{const}.$$

putting in (2) to get the solution as

$$u(x,y) = -\frac{(y+1)^2}{x^3} = c = \text{const}.$$

check

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

$$\Rightarrow du = \frac{3(y+1)^2}{x^4}dx - \frac{2(y+1)}{x^3}dy = 0$$

we get

$(\frac{2}{x}+1)dx + \frac{2}{y}dy = 0$

Let $M = \frac{2}{x}+1$, $N = 2/y$

$\Rightarrow \frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x} = 0$ hence it is the integrating factor.

Now $u = \int M dx \Rightarrow u = \int(\frac{2}{x}+1)dx$.

$\Rightarrow u = 2\ln x + x + k(y)$ —— ②

$\Rightarrow \frac{\partial u}{\partial y} = k(y)$. but $\frac{\partial u}{\partial y} = 2/y = \frac{dk(y)}{dy}$.

$\therefore \frac{dk}{dy} = 2/y \Rightarrow \int \frac{dk}{dy} dy = \int 2/y \, dy$.

$\Rightarrow k = 2\ln y + \ln c$ putting in ② to get the solution as

$u(x,y) = 2\ln x + x + 2\ln y = \ln c$

or $\ln x^2 + \ln y^2 + x = \ln c$.

$\Rightarrow \ln \frac{x^2y^2}{c} = -x \Rightarrow \frac{x^2y^2}{c} = e^{-x}$

$\Rightarrow x^2y^2e^x = c$ Ans

(30) $2\cos y\, dx = \tan 2x \sin y\, dy$, $\cos 2x$

$\Rightarrow 2\cos y\, dx - \tan 2x \cdot \sin y\, dy = 0$

Multiplying with $\cos 2x$

$\Rightarrow 2\cos 2x \cos y\, dx - \cos 2x \tan 2x \cdot \sin y\, dy = 0$.

$\Rightarrow 2\cos 2x \cos y\, dx - \sin 2x \sin y\, dy = 0$.

Now let $M = 2\cos 2x \cos y$ & $N = -\sin 2x \sin y$.

$\Rightarrow \frac{\partial M}{\partial y} = -2\cos 2x \sin y = \frac{\partial N}{\partial x} = -2\cos 2x \sin y$.

Since the equation becomes **exact**. Hence it is the integrating factor.

now solve the Remaining part of the the Question to get the solution yourself.

Let $M = 2\cosh x \cos y$ & $N = -\sinh x \sin y$, so that

$\frac{\partial M}{\partial y} = -2\cosh x \sin y$ & $\frac{\partial N}{\partial x} = -\cosh x \sin y$.

Q3) Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, hence the equation is not exact.

In order to make it exact we have to find its integrating factor.

$$I.F = \exp \int \frac{1}{N}(M_y - N_x)\,dx$$

$$\Rightarrow I.F = \exp \int \left(\frac{-2\cosh x \sin y + \cosh x \sin y}{-\sinh x \sin y} \right) dx$$

$$\Rightarrow I.F = \exp \int \frac{-\cosh x \sin y}{-\sinh x \sin y}\, dx$$

$$\Rightarrow I.F = \exp \int \coth x \, dx$$

$$\Rightarrow I.F = \exp . \ln(\sinh x) = e^{\ln(\sinh x)} = \sinh x.$$

Now multiplying this with (1).

(1) $\Rightarrow 2\sinh x \cosh x \cos y \, dx - \sinh^2 x \sin y \, dx$.

so that $\frac{\partial M}{\partial y} = -2\sinh x \cosh x \sin y$ & $\frac{\partial N}{\partial x} = -2\sinh x \cosh x \sin y$

as $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the equation is now exact.

Now

$$u = \int M dx \Rightarrow u = \int 2\sinh x \cosh x \cos y \, dx.$$

$$\Rightarrow u = \cos y \int 2\sinh x \cosh x \, dx$$

$$= 2\cos y \int f(x) \cdot f'(x)\, dx$$

$$\Rightarrow u = \cos y \sinh^2 x + k(y)$$

$\Rightarrow k = \text{const}$ putting in (2) to get the general solution as:

$u(x,y) = \cos y \sinh^2 x = c = \text{const}$

Check

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$\Rightarrow du = 2\cos y \sinh x \cosh x \, dx + \sinh^2 x(-\sin y)dy$

Ans

(32) $2xy\,dx + 3x^2 dy = 0$ —— (1)

For I.F we have the formula

$I.F = \exp\int \frac{1}{N}(M_y - N_x)dx$

but we can also find the I.F by inspection

multiply (1) with $\frac{1}{x^2y}$

$\Rightarrow \frac{2xy}{x^2y}dx + \frac{3x^2}{x^2y}dy = 0$

$\Rightarrow \underbrace{\frac{2}{x}}_{M}dx + \underbrace{\frac{3}{y}}_{N}dy = 0$ exact $\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x} = 0$

Now for the sol.

$u = \int \frac{2}{x}dx \Rightarrow u = 2\ln x + k(y)$

$\Rightarrow u = 2\ln x + ky$ —— (2)

$\Rightarrow \frac{\partial u}{\partial y} = \frac{dk}{dy} \Rightarrow \frac{\partial u}{\partial y} = N = \frac{3}{y} = \frac{dk}{dy}$

$\Rightarrow \int \frac{dk}{dy}dy = \int \frac{3}{y}dy$

$\Rightarrow k = 3\ln y + c \Rightarrow$ putting in (2) to get the solution as:

$u(x,y) = 2\ln x + 3\ln y = c$

or $\ln x^2 + \ln y^3 = c$ —— (3)

$\Rightarrow x^2y^3 = e^c \Rightarrow x^2y^3 = c$ —— (4) Ans

(33) $(2\cos y + 4x^2)dx = x\sin y\, dy$

$\Rightarrow (2\cos y + 4x^2)dx - x\sin y\, dy = 0$ —— (1)

$\Rightarrow I.F = \exp\int \frac{-2\sin y + \sin y}{-x\sin y}dx = \exp\int \frac{+\sin y}{+x\sin y}dx$

$\Rightarrow I.F = \exp(+\ln x) = x^{+1} = x = \frac{1}{x^{-1}}$

multiplying with (1)

$\Rightarrow \underbrace{\frac{2\cos y}{x^{-1}} + \frac{4x^2}{x^{-1}}}_{M}dx - \underbrace{\frac{x\sin y}{x^{-1}}}_{N}dy = 0 \Rightarrow 2x\cos y + 4x^3 dx - x^2\sin y\, dy = 0$

$\frac{\partial M}{\partial y} = \frac{-2\sin y}{x^{-1}} = -2x\sin y = \frac{\partial N}{\partial x} = -2x\sin y$

solve yourself.

(34) $2\cos y\, dx - \sin y\, dy = 0$ —— (1)

$I.F = \exp\int \frac{-2\sin y - 0}{-\sin y}dx = \exp\int 2dx = \exp(2x) = e^{2x}$

multiplying with (1)

$\Rightarrow \underbrace{2e^{2x}\cos y\, dx}_{M} - \underbrace{e^{2x}\sin y\, dy}_{N} = 0$

$\frac{\partial M}{\partial y} = -2e^{2x}\sin y = \frac{\partial N}{\partial x} = -2e^{2x}\sin y$

So the equation now becomes exact

next solve yourself.

(35) $2x\tan y\, dx + \sec^2 y\, dy = 0$ —— (1)

$I.F = \exp\int \frac{2x\sec^2 y - 0}{\sec^2 y}dx = \exp\int 2x\, dx = e^{x^2}$

-ing with (1)

$\Rightarrow \underbrace{2xe^{x^2}\tan y\, dx}_{M} + \underbrace{e^{x^2}\sec^2 y}_{N}dy = 0$

$\Rightarrow \frac{\partial M}{\partial y} = 2xe^{x^2}\sec^2 y = \frac{\partial N}{\partial x} = 2xe^{x^2}\sec^2 y$

now the equation is exact.

For solution

$u = \int 2xe^{x^2}\tan y\, dx$

and $x^2 = t \Rightarrow dt = 2x\,dx \Rightarrow \frac{dt}{2} = x\,dx$

$\Rightarrow I = \int \frac{e^t}{2}\,dt = \frac{1}{2}e^t = \frac{1}{2}e^{x^2} + k(y)$. putting in ②

$\Rightarrow u = 2\tan y \cdot (\frac{1}{2}e^{x^2}) + k(y)$

$\Rightarrow u = e^{x^2}\tan y + k(y)$ —— ③

$\Rightarrow \frac{\partial u}{\partial y} = e^{x^2}\sec^2 y + \frac{dk}{dy}$.

but $\frac{\partial u}{\partial y} = e^{x^2}\sec^2 y = N$

$\therefore \frac{\partial u}{\partial y} = e^{x^2}\sec^2 y = e^{x^2}\sec^2 y + \frac{dk}{dy} \Rightarrow k = \text{const}$ putting in ③ we get

$u(x,y) = e^{x^2}\tan y = c = \text{const}$

Ans.

(36) $(y+1)dx - (x+1)dy = 0$ —— ①

$I \cdot F = \exp \int \frac{1+1}{-(x+1)}dx = \int \frac{2}{-(x+1)}dx = e^{-2\int (x+1)^{-1} dx}$

$\Rightarrow I \cdot F = e^{-2\ln(x+1)} = (x+1)^{-2}$ multiplying with ①

$\Rightarrow$

$(x+1)^{-2}(y+1)dx - (x+1)^{-2}(x+1)dy = 0$

$\Rightarrow \frac{(y+1)}{(x+1)^2}dx - (x+1)^{-1}dy = 0$.

this equation is now exact

bcz $\frac{1}{(x+1)^2} = \frac{1}{(x+1)^2}$.

For Solution we have

$u = \int \frac{(y+1)}{(x+1)^2}dx \Rightarrow u = -\frac{(y+1)}{(x+1)} + k(y)$ —— ②

$\Rightarrow \frac{\partial u}{\partial y} = \frac{-1}{(x+1)} + \frac{dk}{dy}$ —— ③

but $\frac{\partial u}{\partial y} = N = \frac{-1}{(x+1)} = \frac{-1}{(x+1)} + \frac{dk}{dy}$

$\Rightarrow k = \text{const}$ putting in ② to get the sol is

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$u(x,y) = -\frac{(y+1)}{(x+1)} = c = \text{const.}$ Ans

(37) $x^{-1}\cosh y\,dx + \sinh y\,dy = 0$ —— (1)

$I.F = \exp\int \frac{+x^{-1}\sinh y - 0}{\sinh y}\,dx = \exp(\ln x) = x.$

multiplying with (1)

Eq $\Rightarrow \underbrace{\cosh y\,dx}_{M} + \underbrace{x\sinh y}_{N}\,dy = 0$ —— (2)

This Eq is now exact because $\frac{\partial M}{\partial y} = \sinh y = \frac{\partial N}{\partial x}$.

For solution we have the formula

$u = \int \cosh y\,dx$.

$\Rightarrow u = x\cosh y + k(y)$ —— (3)

$\Rightarrow \frac{\partial u}{\partial y} = x\sinh y + \frac{dk}{dy}$, but $\frac{\partial u}{\partial y} = N = x\sinh y$.

$\therefore x\sinh y = x\sinh y + \frac{dk}{dy} \Rightarrow k = \text{const}$ putting in (3)

to get the solution as

$u(x,y) = x\cosh y = c = \text{const}$ Ans.

Read Prob (39)

⊛ LINEAR DIFFERENTIAL EQUATION & BERNOULLI EQUATION

A first order differential equation of the form
(1) — $y' + P(x)y = r(x)$ is called as linear.

For a differential equation to be linear it is must that it ~~should~~ can be written as in (1).

or simply, A first order differential equation is said to be linear, if it can be written as

$y' + P(x)y = r(x)$ —— (1)

This eq is linear in the unknown function y and y', whereas p as well as r on the right may be any given functions of x.

If the right side $r(x)$ is zero for all x in the interval in which we consider the equation. ($r(x) = 0$), the the equation is said to be **homogeneous**; otherwise it is said to be non-**homogeneous**.

Let us find a formula for the general solution of (1) in some interval I, assuming that p and r are continuous in I. For the homogeneous equation

$y' + P(x)y = 0 \qquad \Rightarrow r(x) = 0$

$\Rightarrow \frac{dy}{dx} + P(x)y = 0$

$\Rightarrow dy + yP(x)dx = 0 \Rightarrow \frac{dy}{y} + Pdx = 0$

$\Rightarrow \frac{dy}{y} = -Pdx \Rightarrow \ln y = -\int Pdx + c^*$

by taking exponentials on b.s

$\Rightarrow y = ce^{-\int pdx + c}$ —(2) ($c = \pm e^{c^*}$ when $y \neq 0$)

Here we may also take $c = 0$ to obtain the

The Equation:

$$\frac{dy}{dx} + yp(x) = Q(x) \quad \text{--- (1)}$$

whose left member is linear in both the dependent variables and the derivative, is called a linear equation of 1st order.

e.g

$\frac{dy}{dx} + 3xy = \sin x$ is called linear while

$\frac{dy}{dx} + 3xy^2 = \sin x$ is not.

Solution of Nonhomogeneous Linear Equation

Let us now consider the nonhomogeneous equation (1)

$y' + yP(x) = r(x)$.

$$\Rightarrow \frac{dy}{dx} + (yP_x - r_{(x)}) = 0 \Rightarrow dy + (Py - r)\,dx = 0 \quad \text{--- (3)}$$

this is $Pdx + Qdy = 0$, where $P = Py - r$ and $Q = 1$.

Since it is not exact so the Integrating factor is

$$I.F = \exp\int \frac{P_y - Q_x}{1}\,dx = \exp\int P\,dx.$$

Multiplying with (3) to make the eq (3) exact.

$$\Rightarrow e^{\int Pdx}\,dy + (Py - r)e^{\int Pdx}\,dx = 0.$$

$$\Rightarrow e^{\int Pdx}\frac{dy}{dx} + (Py - r)e^{\int Pdx} = 0.$$

$$\Rightarrow e^{\int Pdx}\left(\frac{dy}{dx} + Py\right) = re^{\int Pdx}.$$

$$\Rightarrow e^{\int Pdx}\frac{dy}{dx} + yPe^{\int Pdx} = re^{\int Pdx}.$$

$$\Rightarrow \frac{d}{dx}\left(e^{\int Pdx}\cdot y\right) = re^{\int Pdx}.$$

$$\Rightarrow e^{\int Pdx}\cdot y = \int re^{\int Pdx}\,dx + c$$

$$\Rightarrow y = e^{-\int Pdx}\left[\int re^{\int Pdx}\,dx + c\right] \quad \text{--- (4)}$$

now let $h = \int p\,dx$.

so (4) can be written as,

$$y = e^{-h}\left[\int r e^{h}\,dx + c\right] \quad \text{--- (5)}$$

This represents the general sol of eq (1)

EXAMPLE 1

$$y' - y = e^{2x}$$

here $p = -1$, $r = e^{2x}$, $h = \int p\,dx = \int -1\,dx = -x$

$$\therefore \; y = e^{-h}\left[\int e^{h} r\,dx + c\right] = e^{+x}\left[\int e^{-x} e^{2x}\,dx + c\right]$$

$$\Rightarrow y = e^{x}\left[\int e^{x}\,dx + c\right] = e^{2x} + ce^{x}$$

Ans.

In simpler cases, such as the present, we may not need the general formula (5) but may wish to proceed directly, multiplying the given equation by $e^{h} = e^{-x}$, thus gives,

$$(y' - y)e^{-x} = (ye^{-x})' = e^{2x-x} = e^{x}.$$

Integrating on both sides, we obtain the same results as before.

$$ye^{-x} = e^{x} + c \Rightarrow y = e^{2x} + ce^{x}.$$

Ans

If the integral cannot be integrated by the usual methods of calculus (as often in practice) we may have to use a numerical methods for integrals or for differential equations itself.

(*) BERNOULLI'S EQUATION

(Reduction to Linear Form)

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An equation of the form.

$$\left.\begin{array}{l} y' + (y)(P(x)) = y^{a} Q(x). \\ \text{or} \\ y^{-a}\dfrac{dy}{dx} + y^{1-a} P(x) = Q(x) \end{array}\right\} \text{— ① where } a \neq 0, 1$$

is called Bernoulli's Equation.

This is reduced to the form ; namely.

$$u' + (1-a)Pu = (1-a)Q. \quad \text{where } u = [y(x)]^{1-a}$$

In ① if $a = 0$ or $a = 1$ then the equation becomes linear otherwise non-linear

We now prove the formula for the non-linear equation.

In ① let $y(x)^{1-a} = u(x)$

$\Rightarrow u' = (1-a)y^{-a}\dfrac{dy}{dx}$.

putting $\dfrac{dy}{dx} = y' = y^{a}Q(x) - yP(x)$ from ① we get

$$u' = (1-a)y^{-a}(y^{a}Q(x) - yP(x))$$

$$\Rightarrow u' = (1-a)(Q(x) - y^{1-a}P(x))$$

$$\Rightarrow u' + (1-a)y^{1-a}P(x) = (1-a)Q(x)$$

$$\Rightarrow u' + uP(x)(1-a) = (1-a)Q(x) \quad \because u = y^{1-a}.$$

$$\boxed{\Rightarrow u' + (1-a)Pu = (1-a)Q} \text{ — ②}$$

By ⑤

$$u = e^{-\int p\,dx}\left[\int e^{\int p\,dx} \cdot r\,dx + c\right].$$

where $p = (1-a)P$ & $r = (1-a)Q$.

Sol. $e^{-\ln x} = e^{\ln x^{-1}} \Rightarrow e^{\ln x^{-1}} = x^{-1} = 1/x$

& $e^{-\ln(\sec x)} = \cos x$.

$e^{-\ln(\sec x)} = e^{\ln(\sec x)^{-1}} = (\sec x)^{-1} = \frac{1}{\sec x} = \cos x$.

(2)

(B) General Solutions: In Probs 3-14, find the general solutions of the following differential Equation

(3) $y' - y = 4$

Here we use the formula (4) i.e

$$y = e^{-h}\left[\int e^{h}\cdot r + c\right] \quad \text{—(4)} \quad \text{where } h = \int p\,dx$$

Here $p = -1$, $r = 4$ & $h = \int p\,dx = -x$. putting values in (4)

$$\Rightarrow y = e^{x}\left[\int e^{-x}\cdot 4 + c\right]$$

$$\Rightarrow y = -e^{x}e^{-x}\cdot 4 + ce^{x}$$

$$\Rightarrow y = -4 + ce^{x} \quad \text{Ans.}$$

(4) $y' + 2y = 2.5$

Here $p = 2$, $r = 2.5$, $h = \int 2dx = 2x$. so

$$(4) \Rightarrow y = e^{-2x}\left[\int e^{2x}\cdot 2.5 + c\right] \Rightarrow y = e^{-2x}\left[\frac{e^{2x}}{2}(2.5) + c\right].$$

$$\Rightarrow y = ce^{-2x} + 2.5/2 \quad \text{Ans}$$

(5) $y' + 3xy = 0$

Here $p = 3x$, $r = 0$, $h = \int 3x\,dx = \frac{3}{2}x^2$.

$$\therefore y = e^{-\frac{3}{2}x^2}\left[\int 0\cdot e^{\frac{3}{2}x^2}dx + c\right] = ce^{-(\frac{3}{2})x^2}$$

Ans

Here $P = k$, $r = e^{-kx}$, $h = kx$.

Q7 $\Rightarrow y = e^{-kx}\left[\int e^{kx}\cdot e^{-kx}dx + c\right]$

$\Rightarrow y = e^{-kx}\left[x + c\right]$.

$\Rightarrow y = xe^{-kx} + ce^{-kx}$ Ans

(8) $y' + 4y = \cos x$

Here $P = 4$, $r = \cos x$, $h = 4x$.

$\Rightarrow y = e^{-4x}\left[\int e^{4x}\cos x\,dx + c\right]$

$\Rightarrow y = e^{-4x}\left[I + c\right]$ —(1)

Now $I = \int e^{4x}\cos x\,dx$

$\Rightarrow I = \frac{e^{4x}}{4}\cos x - \int \frac{e^{4x}}{4}(-\sin x)dx$

$\Rightarrow I = \frac{e^{4x}}{4}\cos x + \left(\frac{e^{4x}}{16}\sin x - \frac{1}{4}\int\frac{e^{4x}}{4}\cos x\,dx\right)$.

$\Rightarrow I = \frac{e^{4x}}{4}\cos x + \frac{e^{4x}}{16}\sin x - \frac{1}{16}\int e^{4x}\cos x\,dx$

$\Rightarrow I = \frac{e^{4x}}{4}\cos x + \frac{e^{4x}}{16}\sin x - \frac{1}{16}I$

$\Rightarrow (1 + \frac{1}{16})I = \frac{e^{4x}}{4}\cos x + \frac{e^{4x}}{16}\sin x$

$\Rightarrow \frac{17}{16}I = \frac{e^{4x}}{4}\cos x + \frac{e^{4x}}{16}\sin x$

$\Rightarrow I = \frac{4e^{4x}}{17}\cos x + \frac{e^{4x}}{17}\sin x$

putting in (1)

$\Rightarrow y = e^{-4x}\left[\frac{4}{17}e^{4x}\cos x + \frac{e^{4x}}{17}\sin x + c\right]$

$\Rightarrow y = \frac{4}{17}\cos x + \frac{\sin x}{17} + ce^{-4x}$ Ans

Q9 $\Rightarrow y' = 2y/x + x^2e^{x}$

$\Rightarrow y' - 2y/x = x^2e^{x}$.

Here $P = -2/x$, so $h = \int -2/x\,dx$

$\Rightarrow h = -2\ln x = \ln x^{-2}$

$r = x^2e^{x}$.

$\Rightarrow y = e^{-\ln x^{-2}}\left[\int e^{\ln x^{-2}}\cdot x^2e^{x}dx + c\right]$.

$\Rightarrow y = x^2\left[\int x^{-2}x^2e^{x}dx + c\right]$.

$\Rightarrow y = x^2\left[\int e^{x}dx + c\right]$.

$\Rightarrow y = x^2e^{x} + cx^2$ Ans

(10) $y' + y = e^{-x}\tan x$.

Here $P = 1$ so $h = x$ and $r = e^{-x}\tan x$.

So $y = e^{-x}\left[\int e^{x}e^{-x}\tan x\,dx + c\right]$

$\Rightarrow y = e^{-x}\left[\int\tan x\,dx + c\right]$.

$\Rightarrow y = e^{-x}(\ln(\sec x) + c)$ Ans

(11) $y' = (y-2)\cot x$.

$\Rightarrow y' - y\cot x = -2\cot x$

Here $P = -\cot x$, $h = \ln\operatorname{cosec}x$

So $e^{h} = e^{\ln\operatorname{cosec}x} = \operatorname{cosec}x$.

& $e^{-h} = e^{-\ln\operatorname{cosec}x} = \sin x$

$r = -2\cot x$

$\therefore y = \sin x\left[\int -2\operatorname{cosec}x\cot x\,dx + c\right]$

$\Rightarrow y = \sin x\left[2\operatorname{cosec}x + c\right]$.

$\Rightarrow y = 2 + c\sin x$ Ans

$\Rightarrow y' + 3y/x = 1/x^4$

Here $P = 3/x$ so $h = 3\ln x$.

and $e^h = x^3$, $e^{-h} = x^{-3}$.

$r = 1/x^4$

$\therefore y = x^{-3}\left[\int x^3 \frac{1}{x^4}\,dx + c\right]$

$\Rightarrow y = x^{-3}\left[\ln x + c\right]$ Ans

(13) $y' + y\sin x = e^{\cos x}$.

Here. $P = \sin x$ hence $h = -\cos x$.

$e^h = e^{-\cos x}$ & $e^{-h} = e^{\cos x}$.

$r = e^{\cos x}$.

$\therefore y = e^{\cos x}\left[\int e^{-\cos x} e^{\cos x}\,dx + c\right]$

$\Rightarrow y = e^{\cos x}\left[x + c\right]$. Ans

(14) $x^2y' + 2xy = \sinh 5x$.

$\Rightarrow y' + 2y/x = \frac{1}{x^2}\sinh 5x$

Here $P = 2/x$ so $h = 2\ln x$

$\Rightarrow e^h = x^2$ and $e^{-h} = x^{-2}$.

& $r = \frac{1\sinh 5x}{x^2}$

$\therefore y = x^{-2}\left[\int \frac{x^2\sinh 5x}{x^2} + c\right]$

$\Rightarrow y = x^{-2}\left[\int \sinh 5x + c\right]$.

$\Rightarrow y = x^{-2}\left[\frac{\cosh 5x}{5} + c\right]$ Ans

- Solve the following Initial value problems.

(15) $y' + 4y = 20$, $y(0) = 2$

Here. $P = 4$, $h = 4x$.

$e^h = e^{4x}$, $e^{-h} = e^{-4x}$

$r = 20$.

$\therefore y = e^{-4x}\left[\int e^{4x}\cdot 20\,dx + c\right]$

$\Rightarrow y = e^{-4x}\left[\frac{20e^{4x}}{4} + c\right]$.

$\Rightarrow y = e^{-4x}\left[5e^{4x} + c\right]$

$\Rightarrow y = 5 + ce^{-4x}$ (general sol) —(1) Ans

Now $y(0) = 2$.

(1) $\Rightarrow 2 = 5 + c$

$\Rightarrow c = -3$ putting in (1)

$\Rightarrow y = 5 - 3e^{-4x}$

Ans

(16) $y' - (1 + 3x^{-1})y = x + 2$, $y(1) = e - 1$

Here. $P = -(1 + 3x^{-1})$

$\Rightarrow h = -3\ln x - x$

$\therefore e^h = x^3$, $e^{-h} = x^{-3}$

$r = x + 2$

$\therefore y = x^{-3}\left[\int x^3(x+2)\,dx + c\right]$

$\Rightarrow y = x^{-3}\left[\int (x^4 + 2x^3)\,dx + c\right]$

$\Rightarrow y = x^{-3}\left[\frac{x^5}{5} + \frac{x^4}{2} + c\right]$

$\Rightarrow y = \frac{x^2}{5} + \frac{x}{2} + cx^{-3}$

Ans

Here $P = -2\tanh(2x) \Rightarrow h = \int P\,dx = \int -2\tanh(2x)\,dx$

Q17 $\Rightarrow h = -\ln\cosh(2x)$ so $e^{h} = \cosh^{-1}(2x) = \operatorname{sech}(2x)$

& $e^{-h} = \cosh(2x)$.

$r = -2\tanh(2x)$ sechax

$\therefore y = \cosh(2x)\left[\int -\operatorname{sech}(2x)\cdot 2\tanh(2x)\,dx + c\right]$

$\Rightarrow y = \cosh 2x\left[\int -2\operatorname{sech}^2(2x)\sinh(2x)\,dx + c\right]$.

$\Rightarrow y = \cosh(2x)\left(-2(\tanh(2x)\sinh(2x) - \int \tanh(2x)\cosh(2x)\,dx) + c\right)$

$\Rightarrow y = \cosh(2x)\left(-2\tanh(2x)\sinh(2x) + 2\int \sinh(2x)\,dx + c\right)$.

$\Rightarrow y = \cosh(2x)\left(-2\tanh(2x)\sinh(2x) + 2\cosh(2x) + c\right)$ — ①

now $y(0) = 4$

$\Rightarrow 4 = \cosh 0(0 + 2\cosh 0 + c)$.

$\Rightarrow 4 = 1 + c \Rightarrow c = 3$. putting in ①

$\Rightarrow y = (-\cosh 2x)(2\tanh 2x)(\sinh 2x) + 2\cosh^2 2x + 3\cosh 2x$

$\Rightarrow y = -2\sinh^2 2x + 2\cosh^2 2x + 3\cosh 2x$

$\Rightarrow y = 2 + 3\cosh 2x$

Ans

⑱ $y' - (1 + 3x^{-1})y = x + 2, \quad y(1) = e - 1$

$\Rightarrow y' - (1 + 3/x)y = x + 2$.

Here $P = -(1 + 3/x) \Rightarrow h = \int P\,dx = -\int (1 + 3/x)\,dx$.

$\Rightarrow h = -\{x + 3\ln x\} \Rightarrow h = -(x + 3\ln x)$

$\therefore e^{h} = e^{-(x + 3\ln x)} = e^{-x}\cdot e^{-3\ln x} = x^{-3}e^{-x}$.

& $e^{-h} = e^{x}e^{3\ln x} = e^{x}x^{3} = x^{3}e^{x}$.

& $r = x + 2$.

according to the formula.

$\therefore y = x^{3}e^{x}\left[\int x^{-3}e^{-x}(x + 2)\,dx + c\right]$.

$\Rightarrow y = x^{3}e^{x}\left[\int (x^{-2}e^{-x} + 2x^{-3}e^{-x})\,dx + c\right]$.

$\Rightarrow y = x^{3}e^{x}\left[\int e^{-x}(x^{-2} + 2x^{-3})\,dx + c\right.$

$$\Rightarrow y = x^3 e^x \left[e^{-x}(-x^{-1} - 4x^{-2}) - \int (-x^{-1} - 4x^{-2}) e^{x} dx + c \right]$$

$y' + 2y = e^x(3\sin 2x + 2\cos 2x)$

$p = 2, \quad r(x) = e^{3x}(3\sin 2x + 2)$.

$h = \int p\,dx = 2x$.

$y = e^{-2x}\left[\int e^{2x}\cdot e^x(3\sin 2x + 2\cos 2x)dx + c\right]$.

$A = 3e^{3x}\sin 2x$.

$= e^{3x}\sin 2x - \int e^{3x}\cdot 2\cos 2x\,dx$

$= e^{3x}\sin 2x - \frac{2e^{3x}}{3}\cos 2x - \frac{4}{3}\int e^{3x}\sin 2x$.

$A + 4/9\,A = \frac{e^{3x}}{3}\left[3\sin 2x + 2\cos 2x\right]$.

$\Rightarrow A = \frac{e^{3x}}{13}\left[3\sin 2x + 2\cos 2x\right]$.

Q2 $y' = 1 + y^2$.

$y' =$

$\frac{dy}{1+y^2} = dx$

y'

$\tan^{-1} y = x + c$

$y - 1 = y^2$

$u = y^2 = y^{-1}$