DIFFERENTIAL EQUATIONS

EXERCISE 1.1

Problems solved by;

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EXAMPLE

$y' = dy/dt$ = rate of change of population w.r.t time

& let y = population present.

From the Question:

$y' = y \Rightarrow \frac{dy}{dt} = y \Rightarrow \int \frac{dy}{y} = \int dt$

$\Rightarrow \ln y = t + \ln c \Rightarrow \ln y - \ln c = t \Rightarrow \ln\left(\frac{y}{c}\right) = t$

$\Rightarrow y = ce^{t}$

Ans

EXAMPLE 4 (BOOK).

$\frac{dy}{dt} = ky \Rightarrow \int \frac{dy}{y} = \int k\,dt \Rightarrow \ln y = kt + \ln c$.

$\Rightarrow \ln(y/c) = kt \Rightarrow y = ce^{kt}$ — (1)

Now from the Question at a certain time $t = 0$, $y = 2$ grams. putting in (1)

$\Rightarrow 2 = ce^{0} \Rightarrow c = 2$.

$\therefore (2) \Rightarrow y_{(t)} = 2e^{kt}$ — (3)

Thus the amount of Radioactive Substance shows exponential decay (exponential decrease with time).

CHECK $y_{(t)} = 2e^{kt}$.

$\Rightarrow \frac{dy}{dt} = 2ke^{kt} \Rightarrow \frac{dy}{dt} = ky \quad \because 2e^{kt} = y$.

& $y(0) = 2e^{0} = 2$.

EXAMPLE 5 (BOOK).

$y' = -y/x$.

$\Rightarrow \frac{dy}{dx} = -y/x \Rightarrow \int \frac{dy}{y} = \int -\frac{dx}{x}$.

$\Rightarrow \ln y = -\ln x + \ln c \Rightarrow \ln y = \ln \frac{c}{x}$.

$\Rightarrow y = c/x$ — (1)

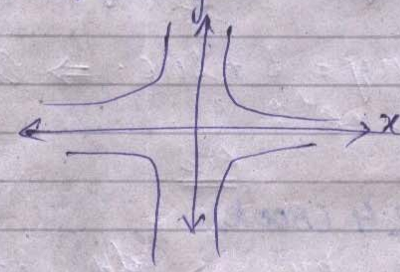
Since we are looking for the curve that passes through $(1,1)$, so we have $y=1$ and $x=1$.

putting in (1):

$\Rightarrow c = 1$.

and hence the particular solution is

$y = 1/x \Rightarrow xy = 1$ (a rectangular hyperbola)



EXERCISE 1.1

Calculus: Solve the following diff equations.

(1) $y' = x^2$

$\Rightarrow \cancel{\int \frac{dy}{dx} = \int x^2} \Rightarrow \frac{dy}{dx} = x^2 \Rightarrow dy = x^2\,dx.$

$\Rightarrow \int dy = \int x^2\,dx \Rightarrow y = \frac{x^3}{3} + c.$

(2) $y' = \sin 3x$

$\Rightarrow y = -\frac{\cos 3x}{3} + c \Rightarrow y = c - \frac{\cos 3x}{3}$

(4) $y' = xe^{-x^2}.$

$\Rightarrow \frac{dy}{dx} = xe^{-x^2} \Rightarrow \int dy = \int xe^{-x^2}\,dx$

$\Rightarrow \cancel{y = x\left(\frac{-e^{-x^2}}{2x}\right) + \int \frac{-e^{-x^2}}{2x}}$

$\Rightarrow \cancel{y = \frac{e^{-x^2} x^2}{2}} -$

put $x^2 = t$

$\Rightarrow 2x\,dx = dt$

$\Rightarrow x\,dx = \frac{dt}{2}$

$\therefore \int y = \int \frac{e^{-t}}{2}\,dt \Rightarrow y = -\frac{1}{2}e^{-t} + c = -\frac{1}{2}e^{-x^2} + c$ Ans.

$y' = -\frac{1}{2}e^{-x^2}(-2x) = xe^{-x^2}$

(5) $y' + y = x^2 - 2$ is ~~1st~~ order diff. equation.

Q5

$\Rightarrow \frac{dy}{dx} + y = x^2 - 2 \Rightarrow dy + y\,dx = (x^2 - 2)dx$

$\Rightarrow \int dy = \int (x^2 - y - 2)dx \Rightarrow y = 2x - y\frac{dy}{dx} + c_1$

$\Rightarrow$

$y = ce^{-x} + x^2 - 2x$

$\Rightarrow y' = -ce^{-x} + 2x - 2$

Now $y' + y = \cancel{-ce^{-x}} + \cancel{2x} - 2 + \cancel{ce^{-x}} + x^2 - \cancel{2x}$

$\Rightarrow y' + y = x^2 - 2$

Verified

→ 2nd order differential Equations

(8) $y'' + y = 0$ — (1), $y = a\cos x + b\sin x$ — (2)

(2) $\Rightarrow y' = -a\sin x + b\cos x$

$\Rightarrow y'' = -a\cos x - b\sin x$

Now $y'' + y = -a\cos x - b\sin x + a\cos x + b\sin x$

$\Rightarrow y'' + y = 0$

verified.

(9) $x + yy' = 0$ — (1); $x^2 + y^2 = 1$ — (2)

~~1st~~ order differential Equations.

(2) $\Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -x/y = y'$

$\Rightarrow$ Now $x + yy' = x + y(-x/y) = 0$

verified.

(10) $x + yy' = 0$, $x^2 - y^2 = 1$

$\Rightarrow 2x - 2y\frac{dy}{dx} = 0$

$\Rightarrow dy/dx = x/y$

$\therefore x + y(x/y) = 2x$ The Equation becomes nonhomogeneous

(11) $x + yy' = 0$, $x^2 - y^2 = 2$

$\Rightarrow 2x - 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = +x/y$.

$\therefore x + y(+x/y) = 2x$ So Nothing happens.

(12) $x^3+y^3y'=0$ —(1)
$x^4+y^4=c$ —(2) $(y>0)$

Q12 $y=1$ when $x=0$

To show that (2) is a sol of (1) differential eq(2) or Integral eq(1)

(2) $\Rightarrow$ $4x^3+4y^3\frac{dy}{dx}=0$

$\Rightarrow x^3+y^3y'=0.$

Hence (2) is a sol of (1)

Now when $x=0,\ y=1$ putting in (2)

$\Rightarrow 0+1=c \Rightarrow c=1$

$\therefore$ The sol is

$x^4+y^4=1.$ Ans

(13) $y'+2y=2.8$ —(1) Q13
$y=ce^{-2x}+1.4$, —(2) $y=1$ when $x=0$

(2) $\Rightarrow y'=-2ce^{-2x}.$

$\Rightarrow y'+2(y-1.4)=0$

$\Rightarrow y'+2y=2.8.$

$\therefore$ (2) is a solution of (1)

Now when $x=0,\ y=1.$

(2) $\Rightarrow 1=c+1.4.$

$\Rightarrow c=-0.4.$ putting in (2)

$\Rightarrow y=-0.4e^{-2x}+1.4$

$\Rightarrow y+0.4e^{-2x}=1.4$

Ans

(14) $xy'=3y$, —(1) $y=cx^3$, —(2) $y=16$ when $x=-4$

(2) $\Rightarrow y'=3cx^2.$

$\Rightarrow xy'=3cx^3.$

$\Rightarrow xy'=3y$ verified

Now when $x=-4,\ y=16.$

$\therefore$ (2) $\Rightarrow 16=c(-4)^3.$

$\Rightarrow c=-4.$

So putting in (2)

$\Rightarrow y=-4x^3 \Rightarrow y+4x^3=0$

Ans

(15) $yy'=2x$, —(1) $y^2-2x^2=c$ —(2) $(y>0)$
$y(1)=\sqrt{3}.$

(2) $\Rightarrow 2yy'-4x=0$

$\Rightarrow yy'-2x=0$

$\Rightarrow yy'=2x$ verified

Now $y(1)=\sqrt{3}.$ putting in (2)

$\Rightarrow (\sqrt{3})^2-2(1)^2=c$

$\Rightarrow 3-2=c$

$\Rightarrow c=1$ putting in (2)

$\Rightarrow y^2-2x^2=1$

Ans

(16) $y'=y\tan x$ —(1) $y=c\sec x$ —(2) & $y(0)=\pi/2.$

(2) $\Rightarrow y'=c\sec x\tan x.$

$\Rightarrow y'=y\tan x$ verified

Now $y(0)=\pi/2$ putting in (2)

$\Rightarrow \pi/2=c\sec 0$

$\Rightarrow c=\pi/2.$ putting in (2)

$\Rightarrow y=\pi/2\sec x.$ Ans

(17) $4yy'+x=0$, —(1) $x^2+4y^2=c$ —(2) $(y>0)$
$y(2)=1$

(2) $\Rightarrow 2x+8yy'=0$

$\Rightarrow x+4yy'=0$ verified

Now $y(2)=1.$

(2) $(2)^2+4(1)^2=c$

$\Rightarrow c=8.$ putting in (2)

$\Rightarrow x^2+4y^2=8$

(18) What happens in Prob.17 if we change initial value condition to $y(a)=0$, where a is a constant.

Sol
$y(a)=0$ putting in (2) (Q#17).
$a+4(0)=c \Rightarrow c=a$.
Q18 $\therefore x^2+4y^2=a$. Ans

Solutions Of Differential Equations.

The problem in diff equations is essentially that of recovering the primitive, that give rise to the differential equation. In other words the problem of solving a differential equation of order n is essentially of finding a relation b/w the variables involving n independent arbitrary constant (Primitive) which together with the derivative obtained from it satisfies the differential equation.

diff Eq **Primitive**

1) $\frac{d^3y}{dx^3}=0$ $y=Ax^2+Bx+C$. — (1)

2) $\frac{d^3y}{dx^3}-6\frac{d^2y}{dx^2}+11\frac{dy}{dx}-6y=0$, $y=c_1e^{3x}+c_2e^{2x}+c_3e^{x}$ — (2)

Particular Solution: of a differential equation is one obtained from the primitive by assigning definite values to the arbitary constants.
e.g in (1) above $y=0$ when $A=B=C=0$,
$y=2x+5$, when $A=0$, $B=2$, $C=5$. are particular solutions.

General Solution:
The primitive of a differential eq is usually called the general sol of the equation.

 Solved by: Umair Asghar

Any 1st order differential equation must contain the 1st derivative y' of the unknown functions y, and it may contain y itself and given function of x. Hence we can write any 1st order differential Eq in the Form.

$$F(x, y, y') = 0 \quad \text{—} (1) \text{ (Implicit Form)}$$

And in most applications (but not always) we can write the differential equation in the explicit Form.

$$y' = f(x, y) \quad \text{———} (2)$$

From Calculus we know that $y' = \frac{dy}{dx}$ = slope of the curve of $y(x)$.

Hence if (2) has a solution $y(x)$ passing through a point (x_0, y_0) of xy-plane, it must have at (x_0, y_0) the slope $f(x_0, y_0)$.

This suggest the idea of plotting approximate solution curves of a given differential equations with actually solving the equation.